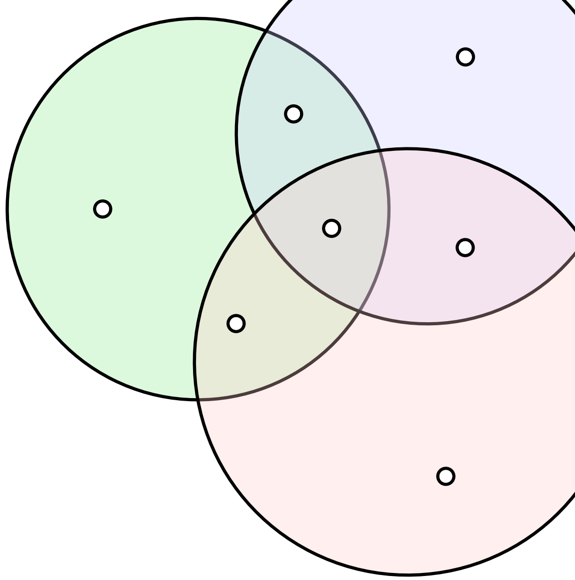
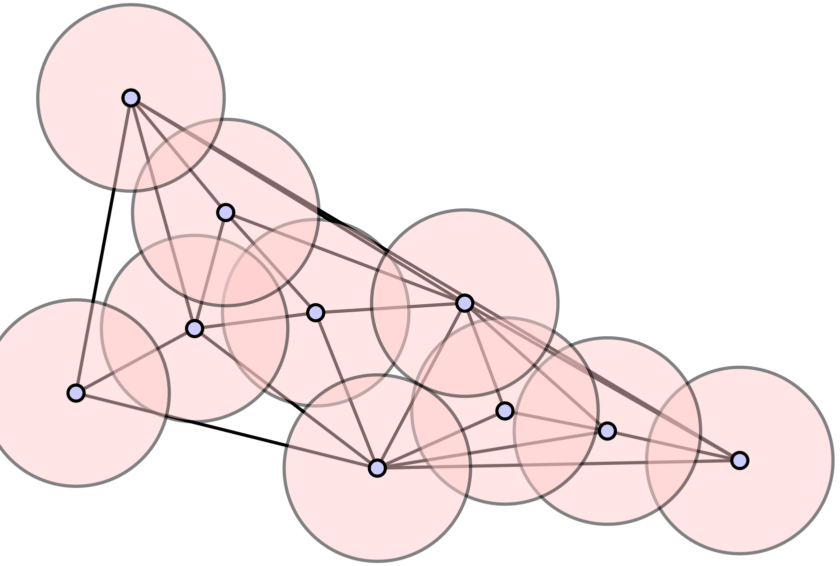
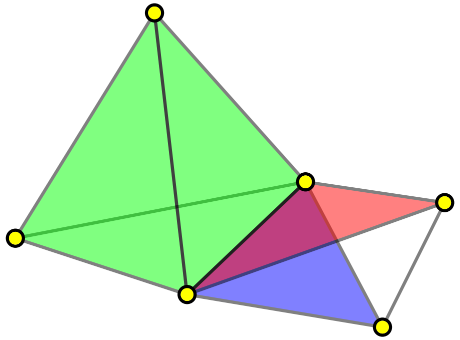


A zest of combinatorial topology applied to the simplification of inclusion-exclusion formulas

Xavier Goaoc   Équipe GAMBLE (LORIA/INRIA)



MidiCombi – Février 2025



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For $X \subset \mathbb{R}^d$ let us write

$$\mathbb{1}_X : \begin{cases} \mathbb{R}^d & \to & \{0,1\} \\ p & \mapsto & \begin{cases} 1 & \text{if } p \in X, \\ 0 & \text{if } p \notin X. \end{cases} \end{cases}$$

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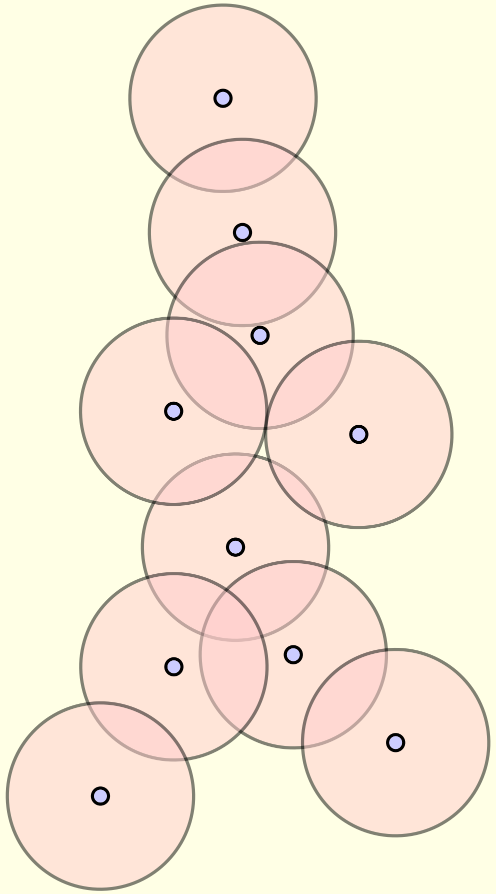

Theorem. [Naiman-Wynn'92]
Let $F = \{b_1, b_2, \ldots, b_n\}$ be a family of equal radius balls in $\mathbb{R}^d$. Letting T denote the Delaunay triangulation of the balls' centers, we have

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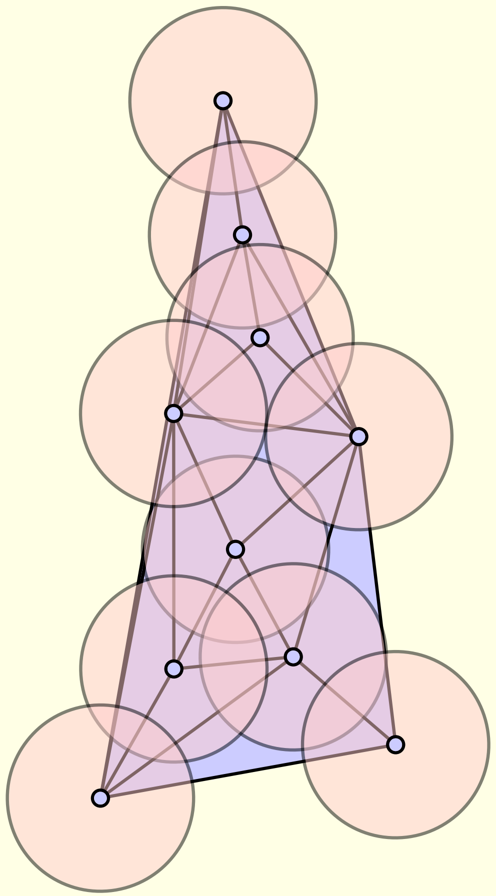

Key ideas

- ▷ Inclusion-exclusion formulas **tailored** to a set system,
- ▷ describe inclusion-exclusion formulas via **abstract simplicial complexes**,
- ▷ interpret IE properties in terms of **Euler characteristic** (χ) of sub-complexes,
- ▷ use the **topological space** associated to a simplicial complex to control its χ.
- ▷ Delaunay $=$ nerve(Voronoi)
- ▷ nerve theorem

Part 2

Theorem. [G-Matoušek-Paták-Safernová-Tancer '15]
Let n and m be integers, and let $D = \lceil 2e \ln m \rceil \left\lceil 2 + \ln \frac{n}{\ln m} \right\rceil$. For every family $F = \{a_1, a_2, \ldots, a_{\mathbf{n}}\}$ of sets with Venn diagram of size **m** there is an abstract simplicial complex $K \subseteq 2^{[n]}$ of dimension $\leq D$ such that

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$O\left(m^{\ln^2 n}\right)$ *summands*

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Could simplified IE help getting this down?

The roadmap...

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doi:10.1017/S096354831400042X

Simplifying Inclusion–Exclusion Formulas

- ▷ A linear-algebraic view
- ▷ A "topological" shortcut
- ▷ An ad hoc model of random simplicial complexes

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Vol. 39, No. 2, pp. 546–563

SET PARTITIONING VIA INCLUSION-EXCLUSION*

ANDREAS BJÖRKLUND†, THORE HUSFELDT†, AND MIKKO KOIVISTO‡

- ▷ Zeta transform and its computation
- ▷ Graph coloring via inclusion-exclusion

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- ▷ What if... ?

#1. A linear-algebraic view

Combinatorics, Probability and Computing: page 1 of 19.
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Simplifying Inclusion–Exclusion Formulas

A **set system** $F = \{a_1, a_2, \ldots, a_n\}$.

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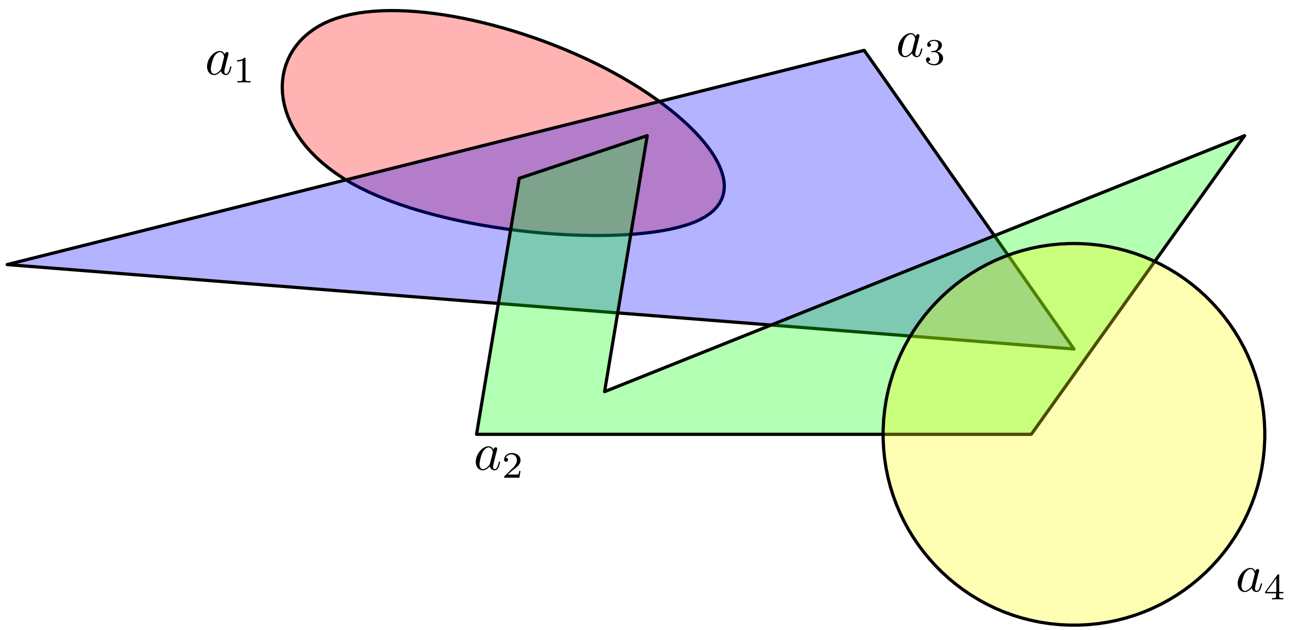

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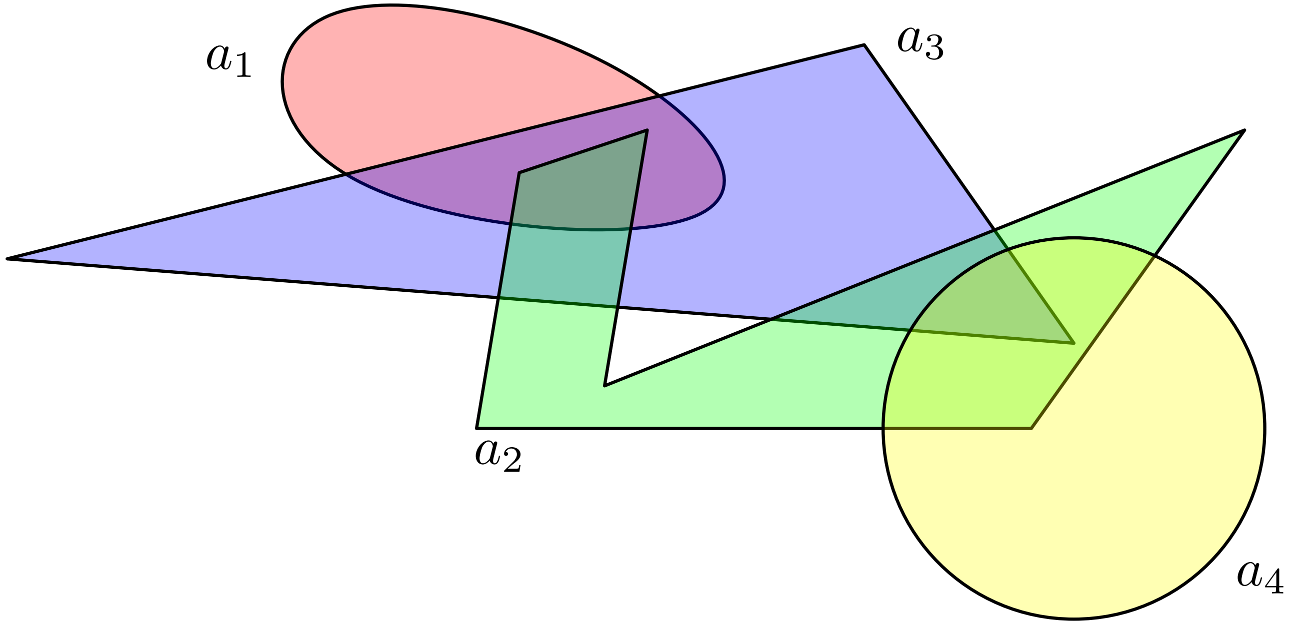

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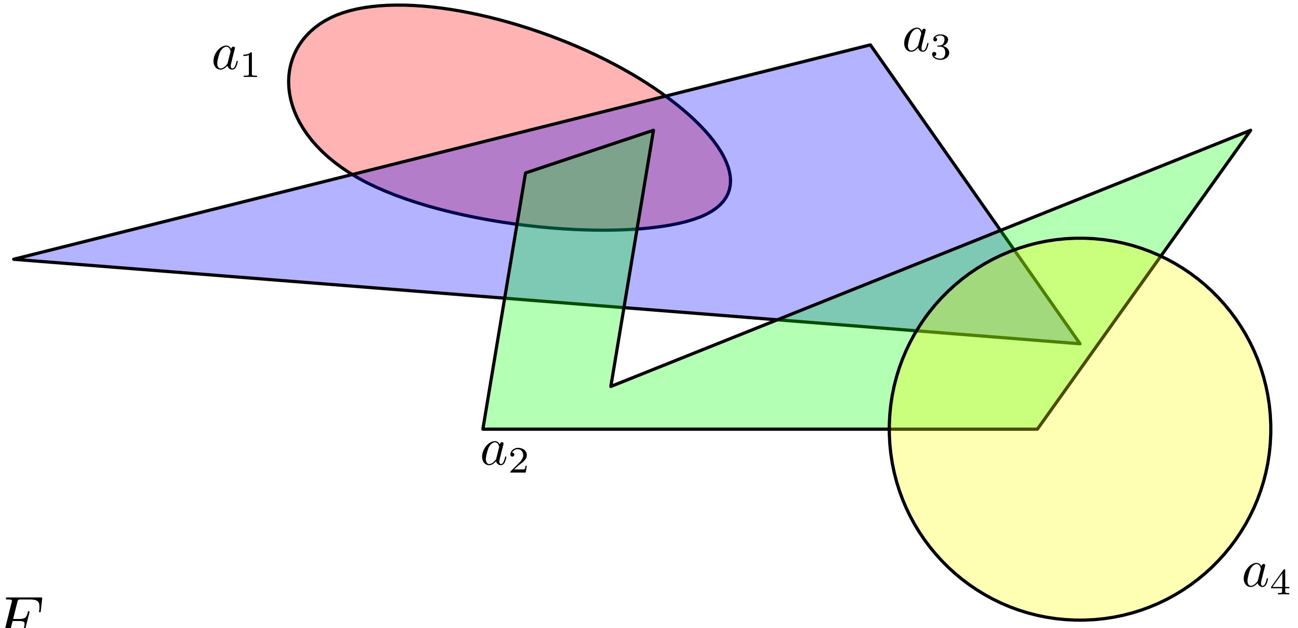

▷ minimal x are supported on the **nerve** of F.

$$N(F) \stackrel{\text{def}}{=} \{I \subseteq [n] \colon \cap_{i \in I} a_i \neq \emptyset\}$$

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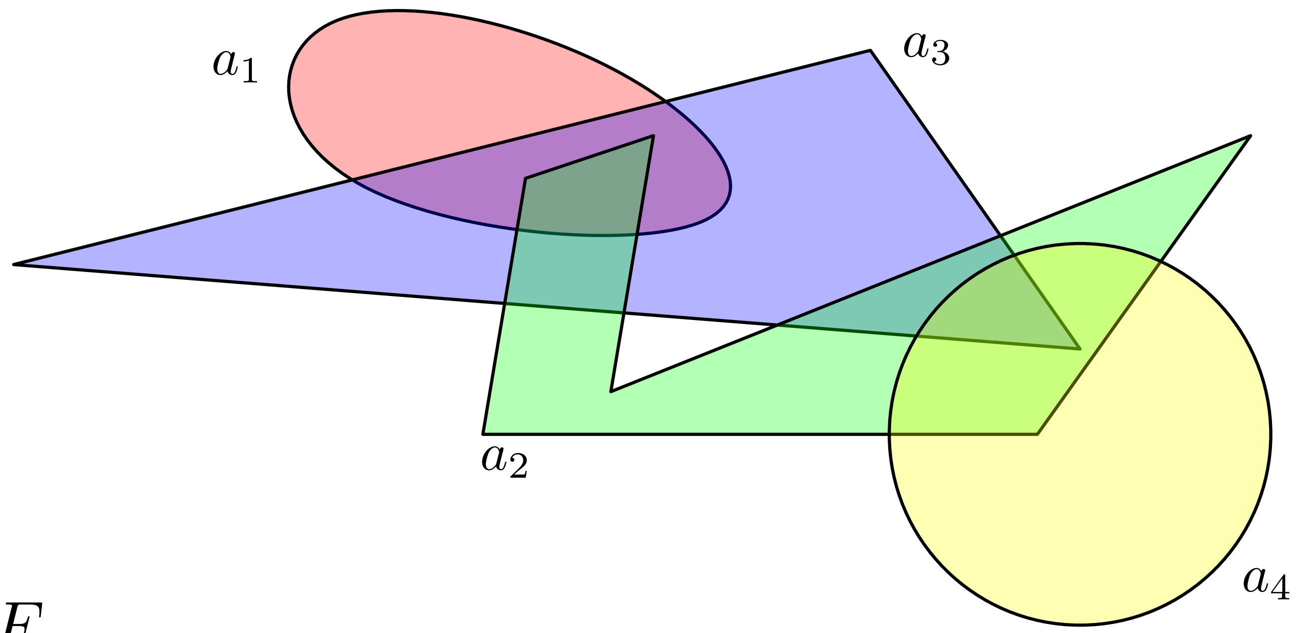

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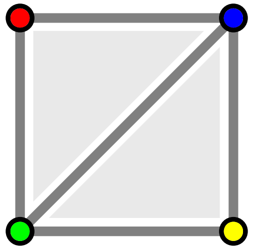

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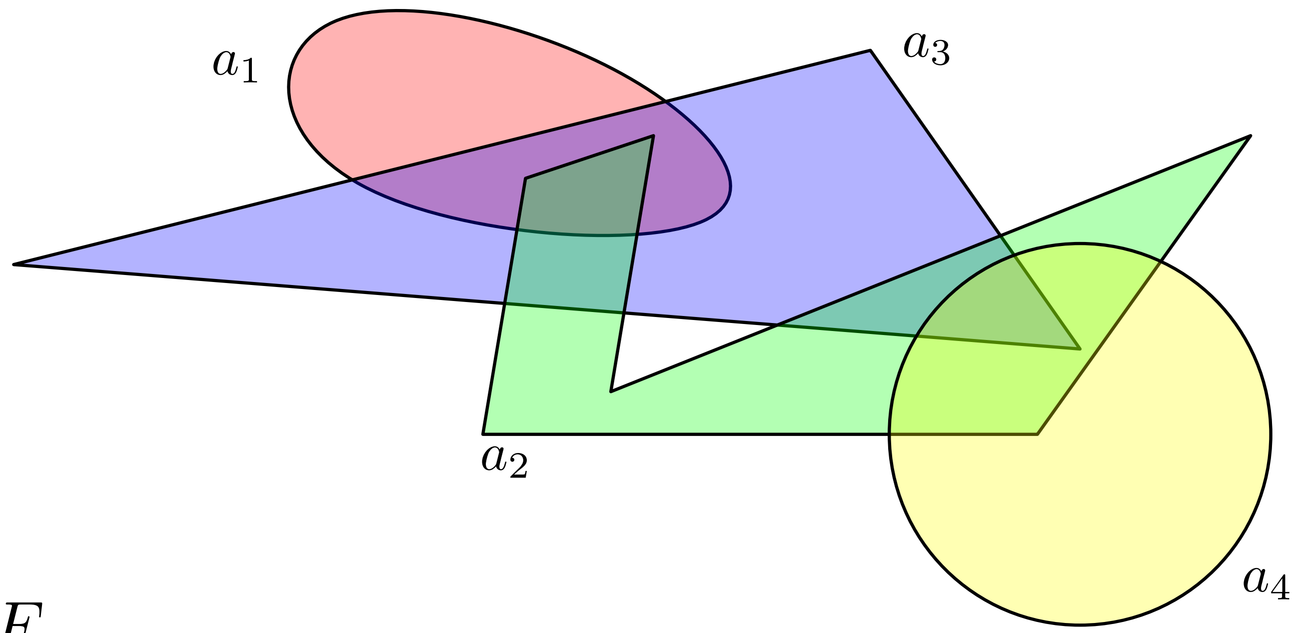

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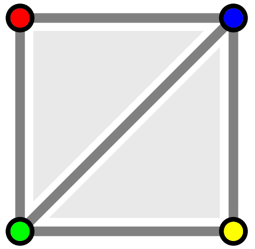

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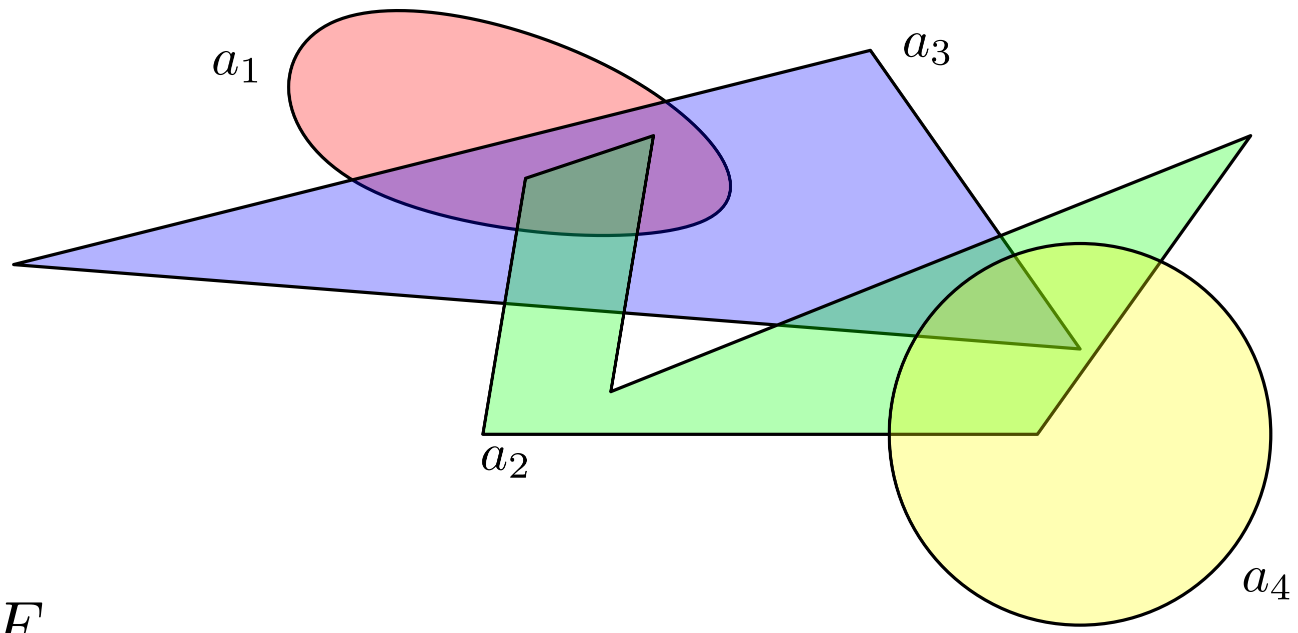

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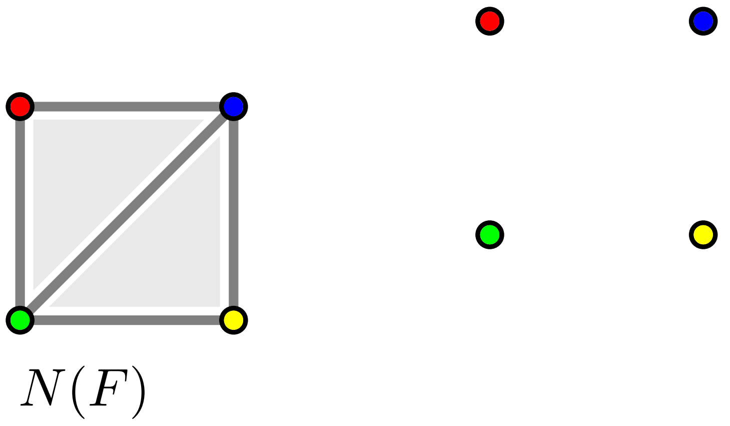

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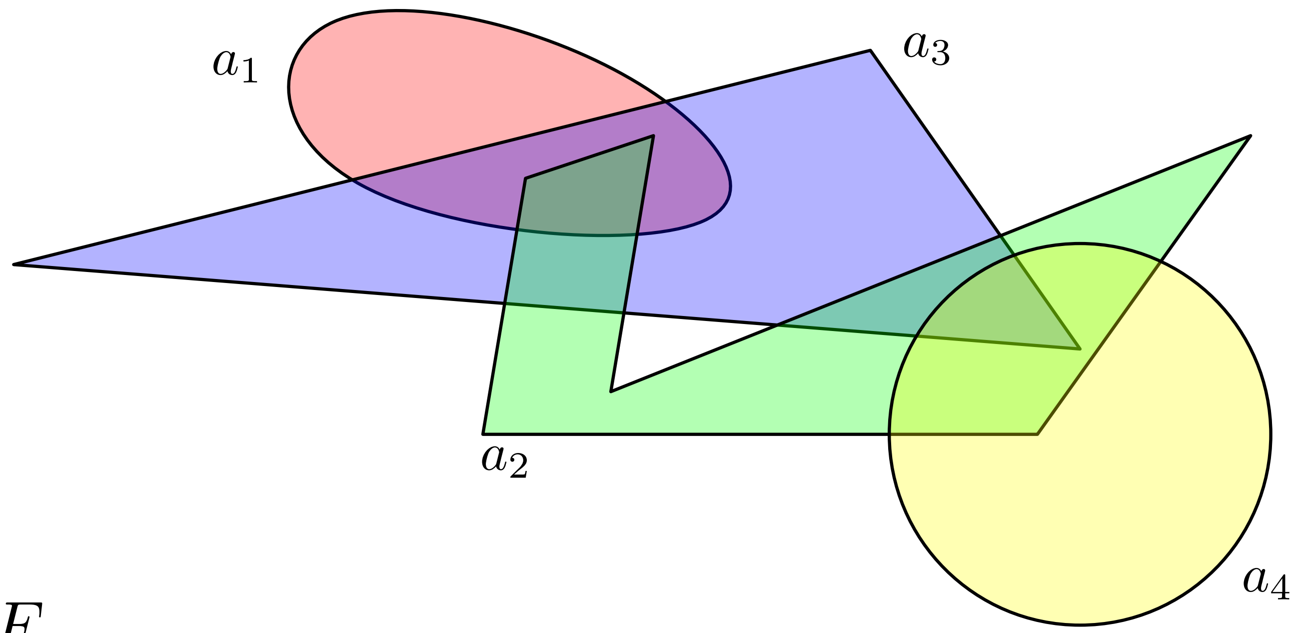

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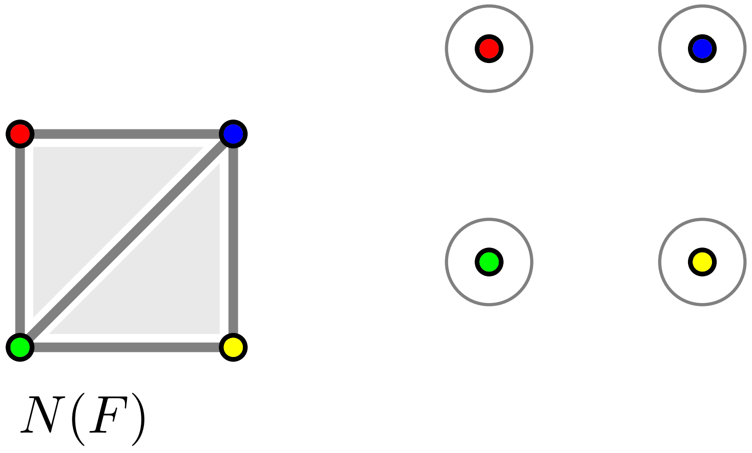

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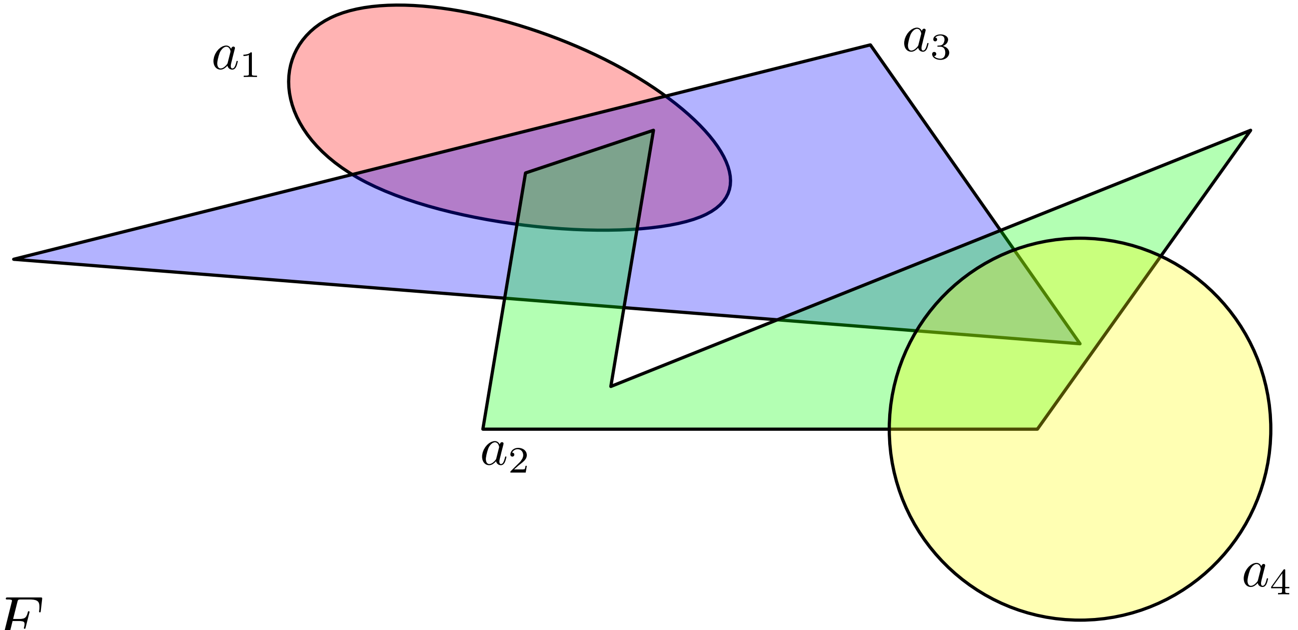

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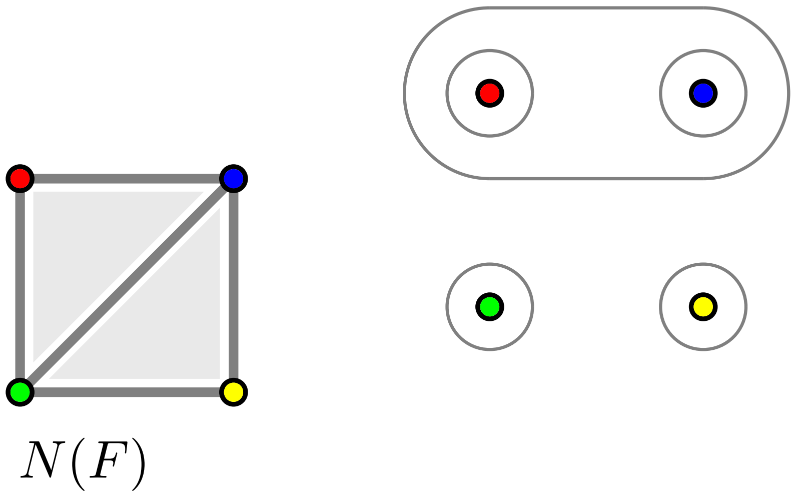

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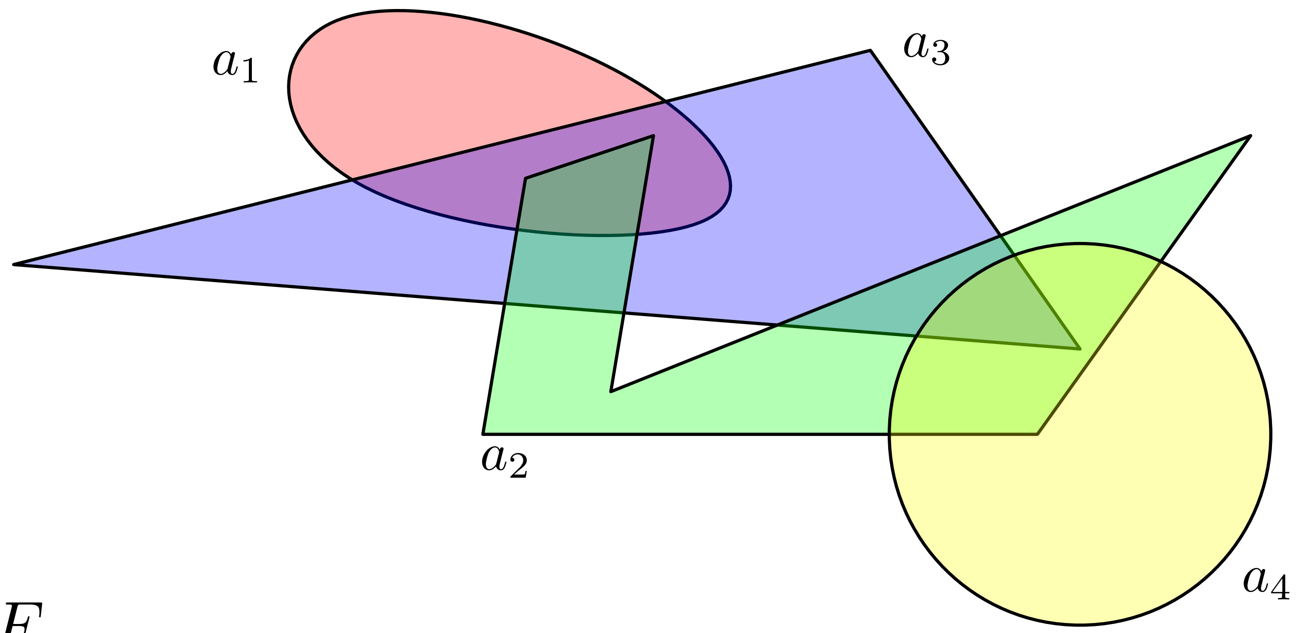

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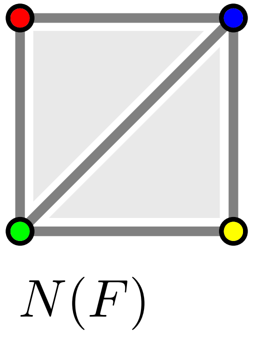

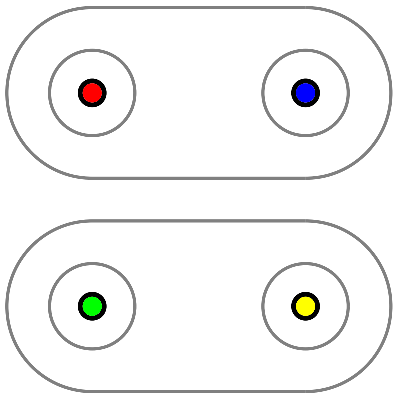

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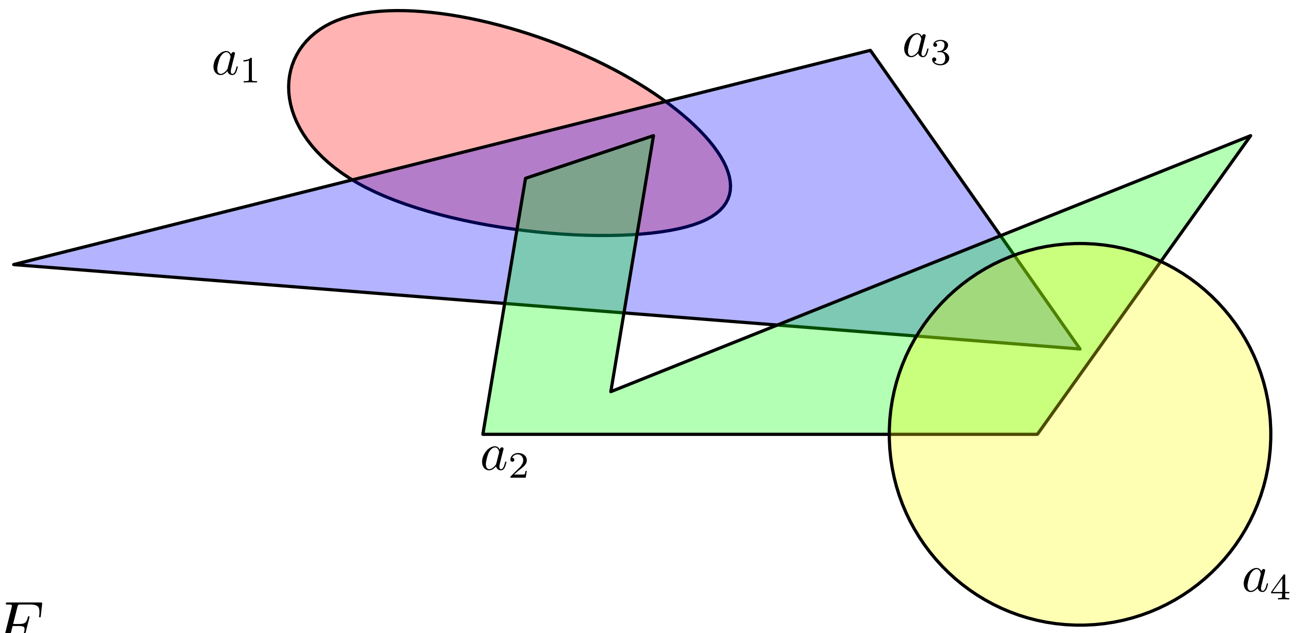

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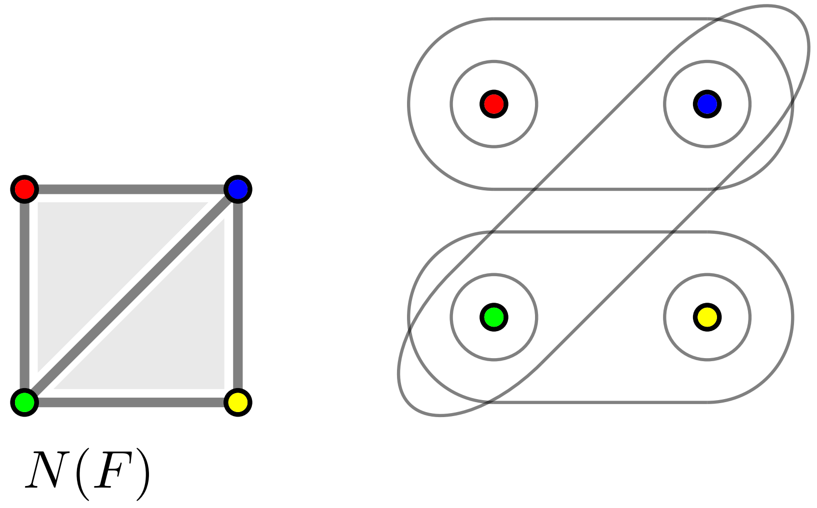

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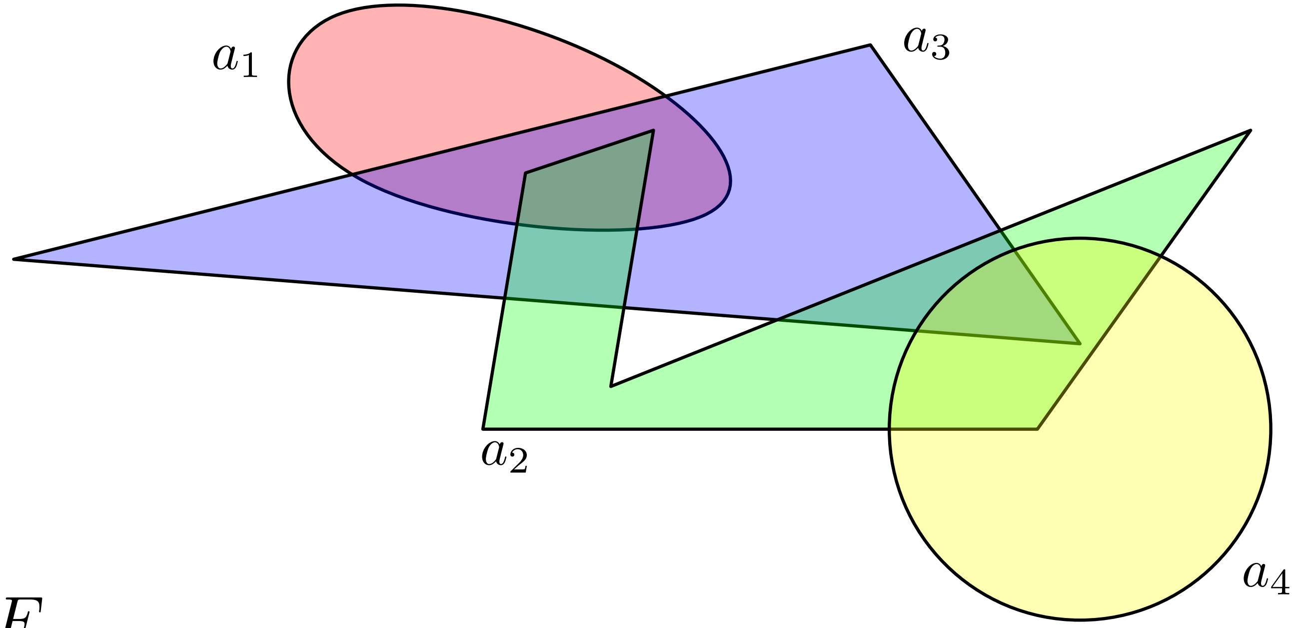

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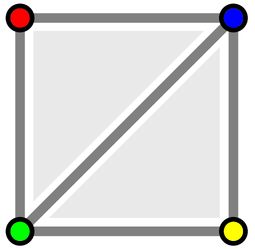

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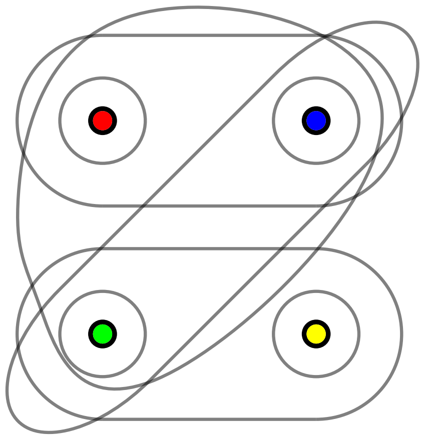

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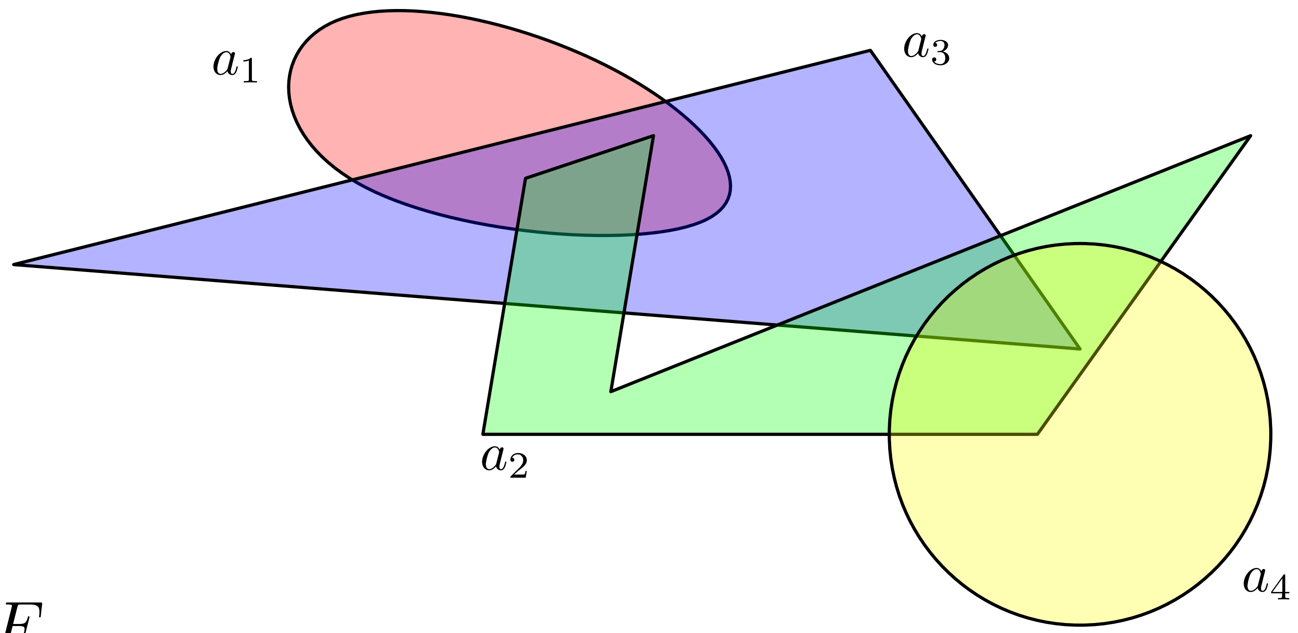

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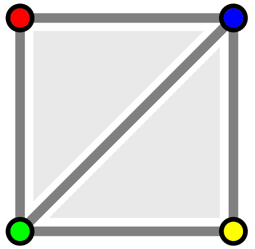

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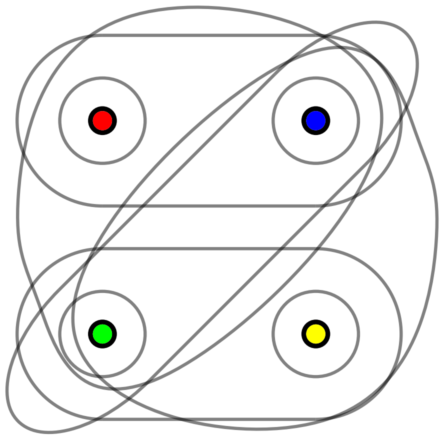

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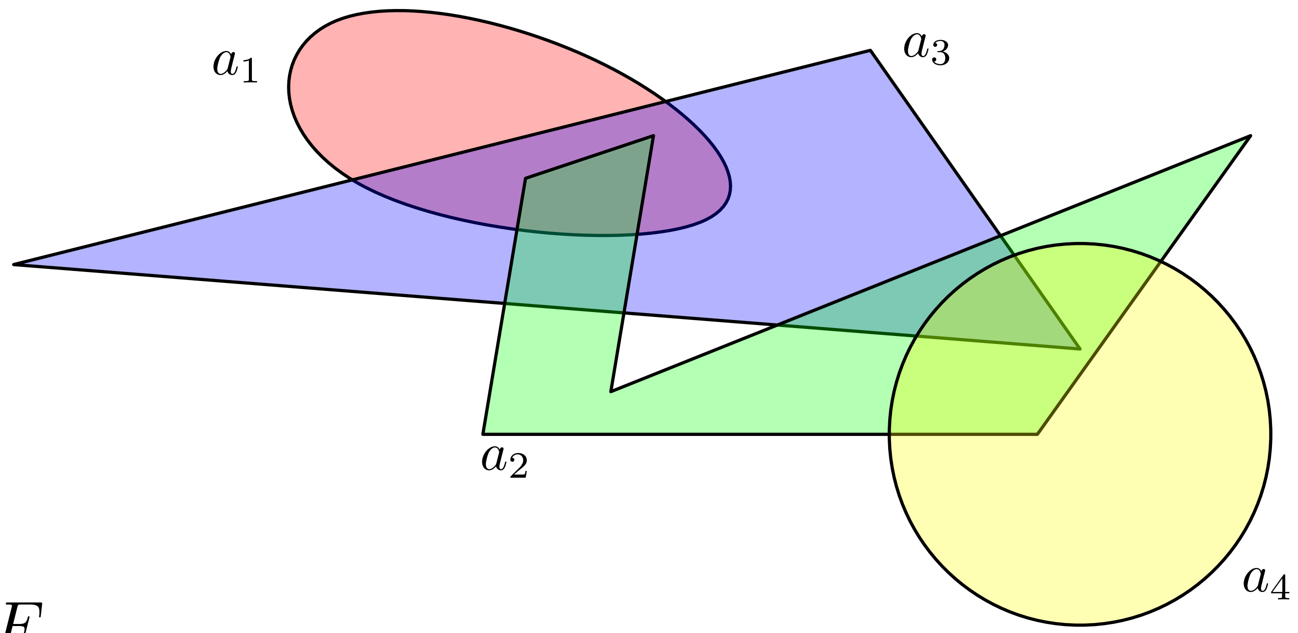

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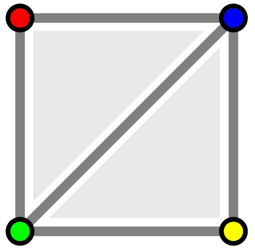
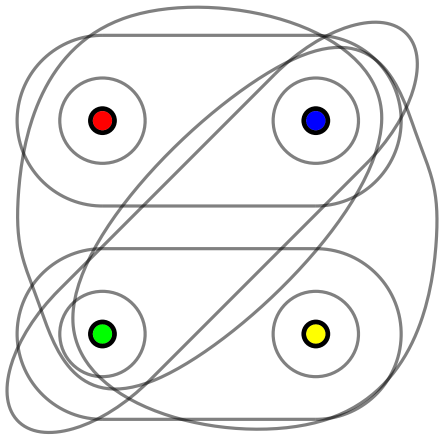
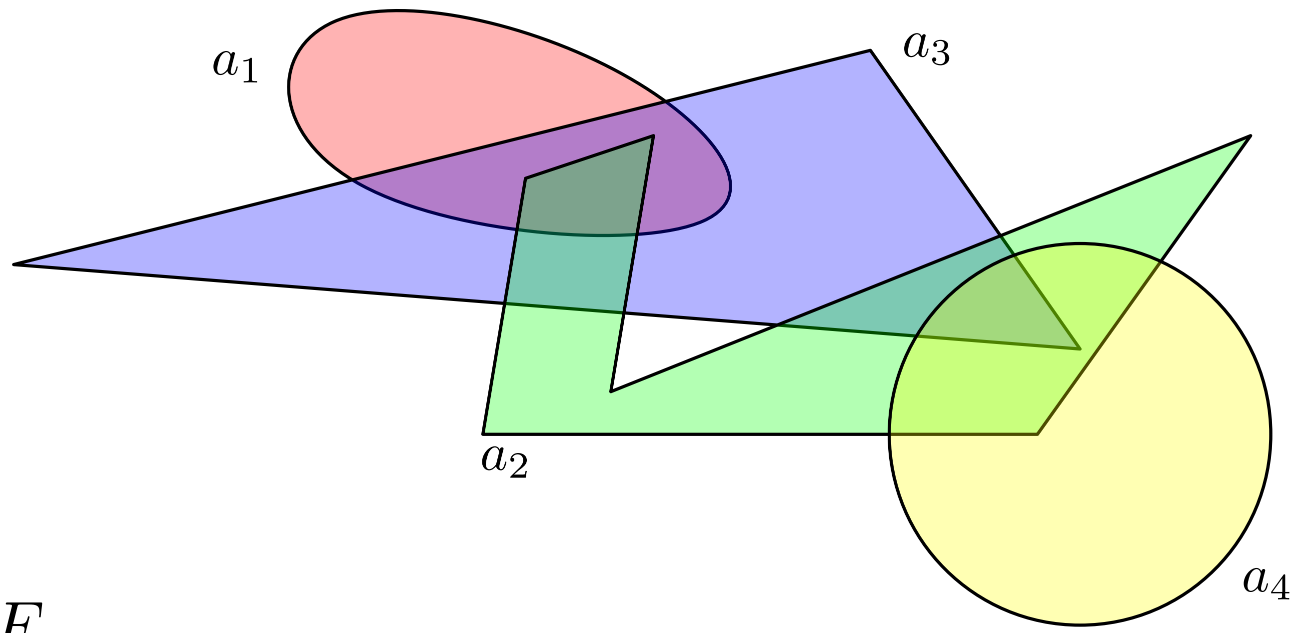


$N(F)$



$V(F)$

A **set system** $F = \{a_1, a_2, \ldots, a_n\}$.



Want: "small" $x \in \mathbb{R}^{(2^{[n]})}$ satisfying

$$\mathbb{1}_{\bigcup_{i=1}^n a_i} = \sum_{I \subseteq [n]} x_I \mathbb{1}_{\bigcap_{i \in I} a_i}$$

- minimal x are supported on the **nerve** of F.

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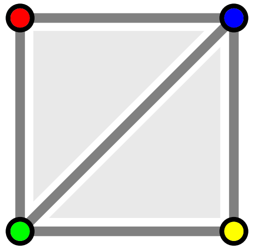
- Inclusion-exclusion formulas for F
  $\leftrightarrow$ solutions of a system of affine equations.



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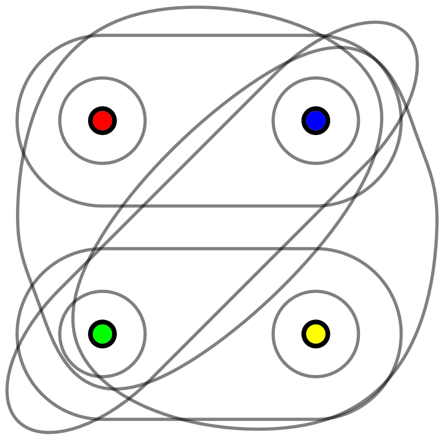

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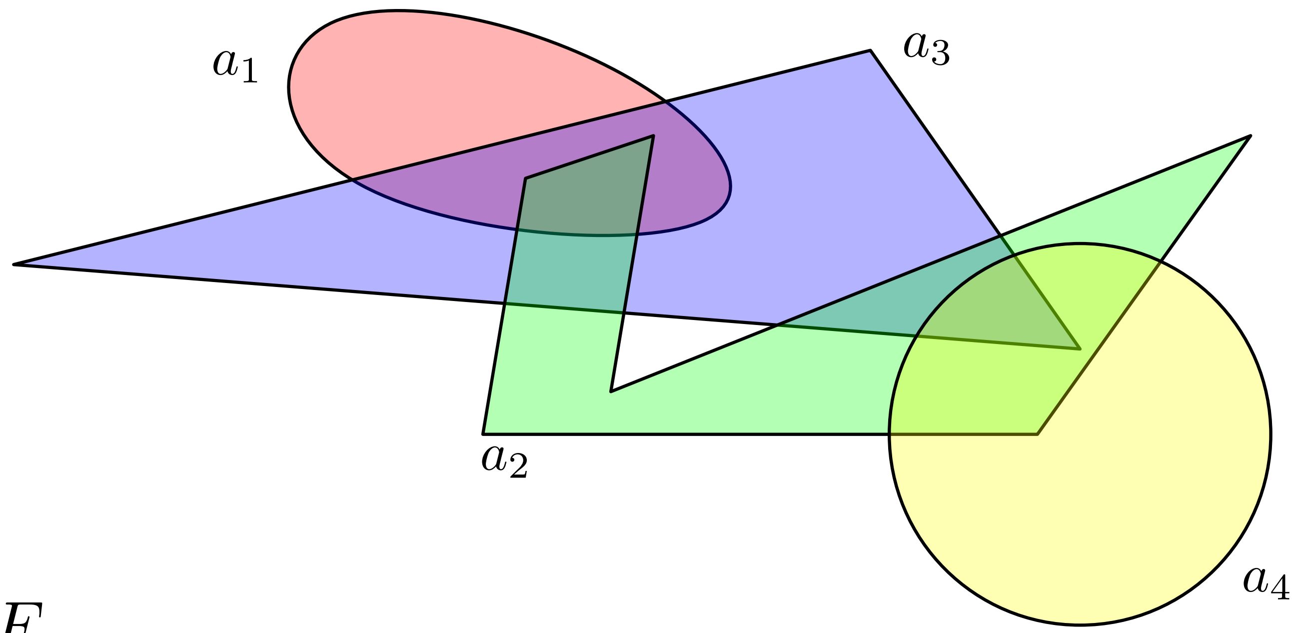

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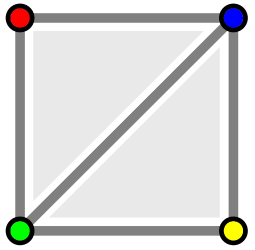
- Inclusion-exclusion formulas for F
  $\leftrightarrow$ solutions of a system of affine equations.

  For each $\tau \in V(F)$, $\displaystyle\sum_{\sigma \subseteq \tau} x_\sigma = 1$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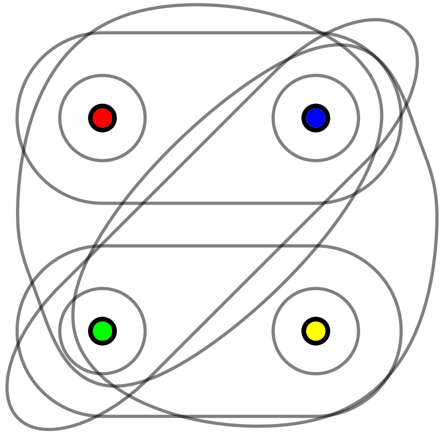
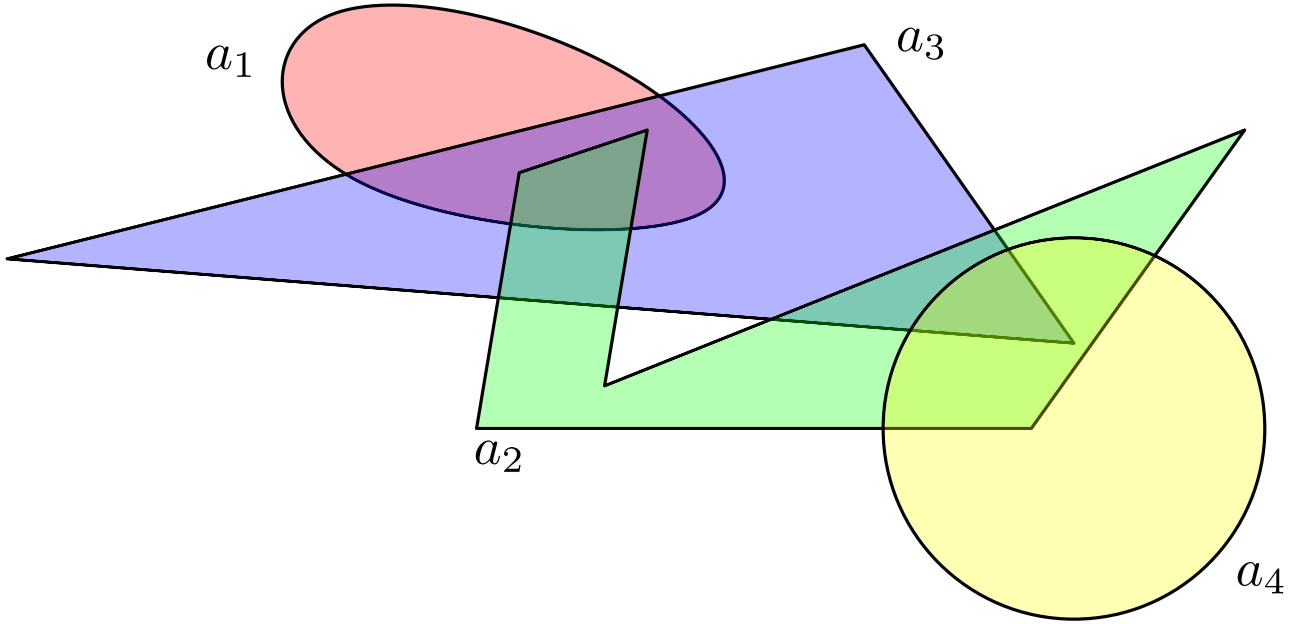

$V(F)$

Write $N(F) = \{\sigma_1, \sigma_2, \ldots, \sigma_N\}$
and $V(F) = \{\tau_1, \tau_2, \ldots, \tau_m\}$



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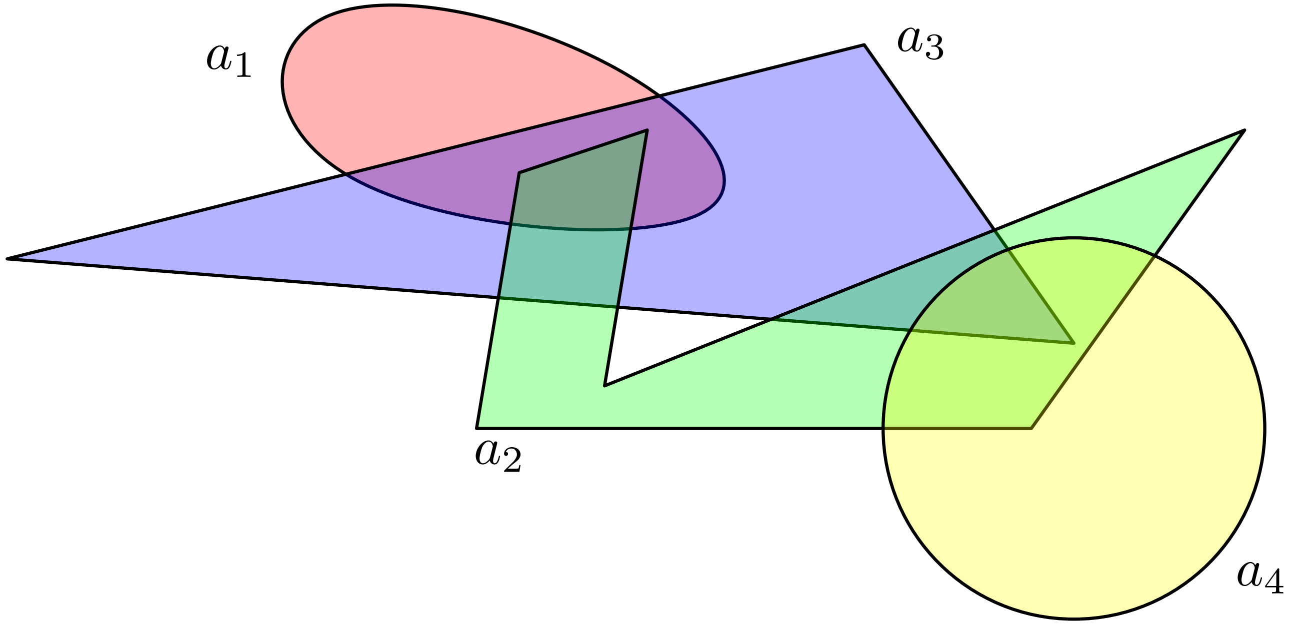

$A \in \{0,1\}^{9 \times 11}$

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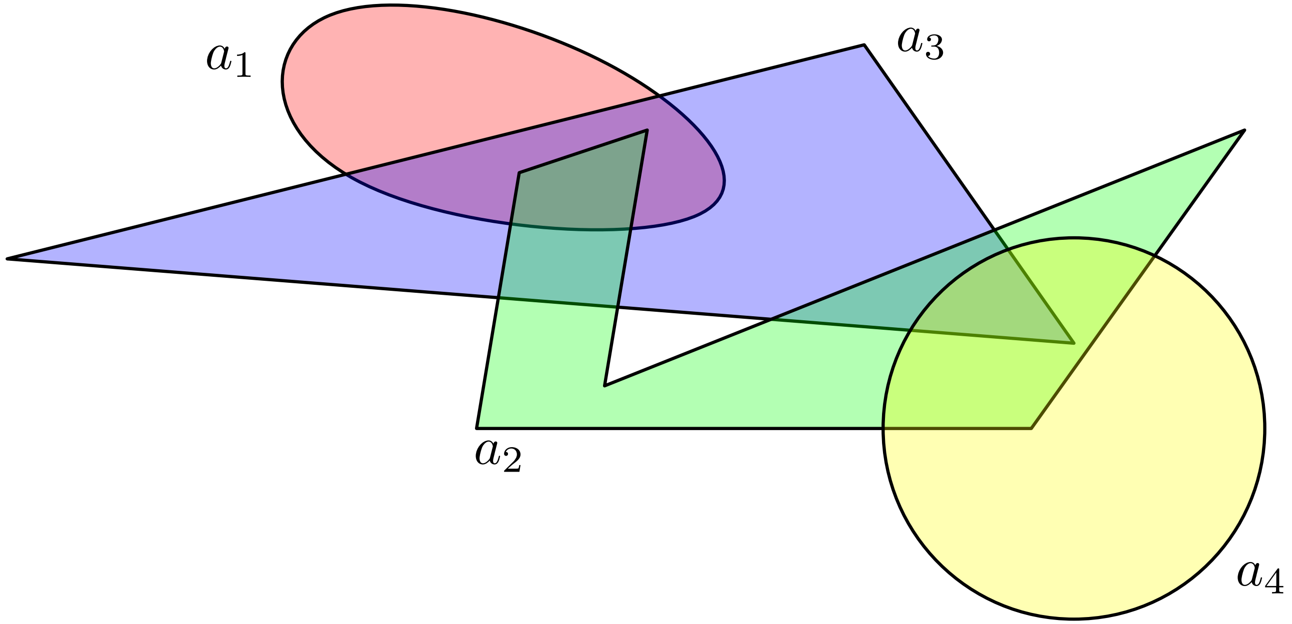

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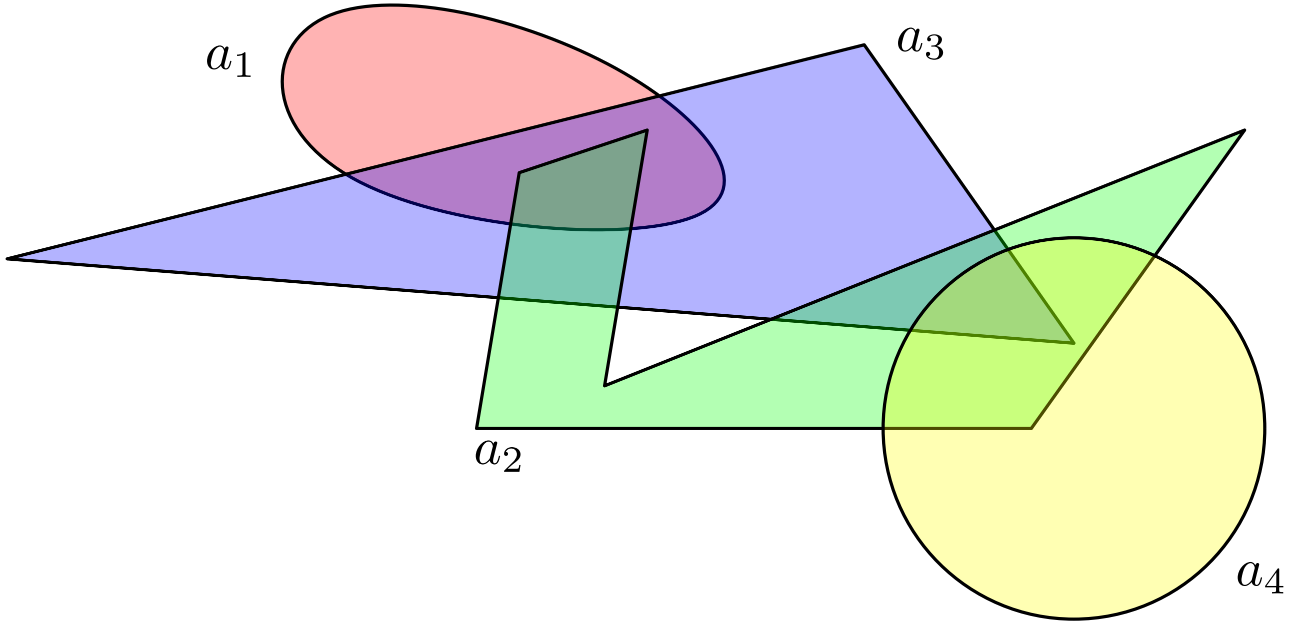

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Integral coordinates, can be exponential in n.

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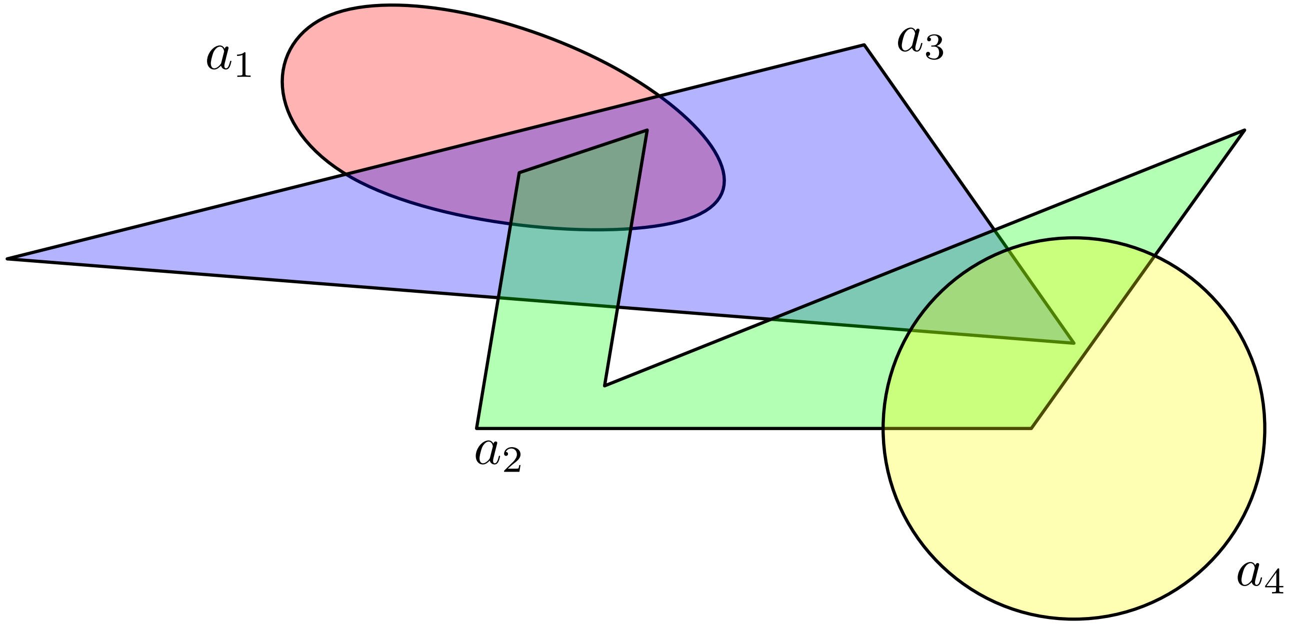

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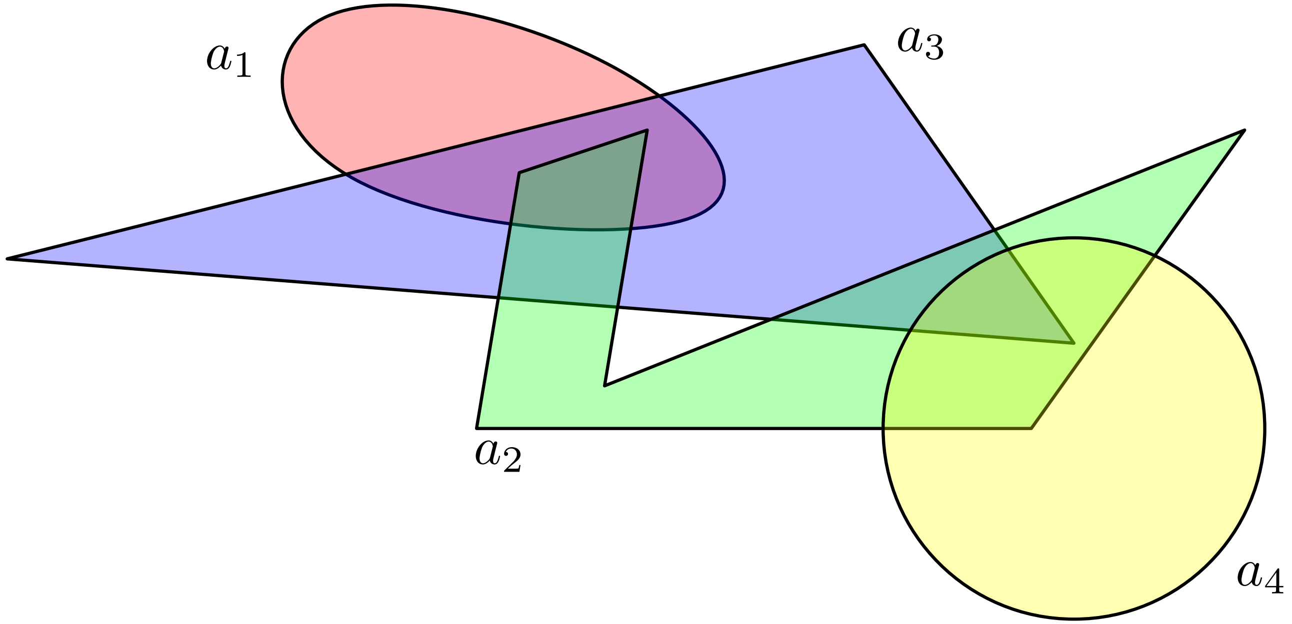

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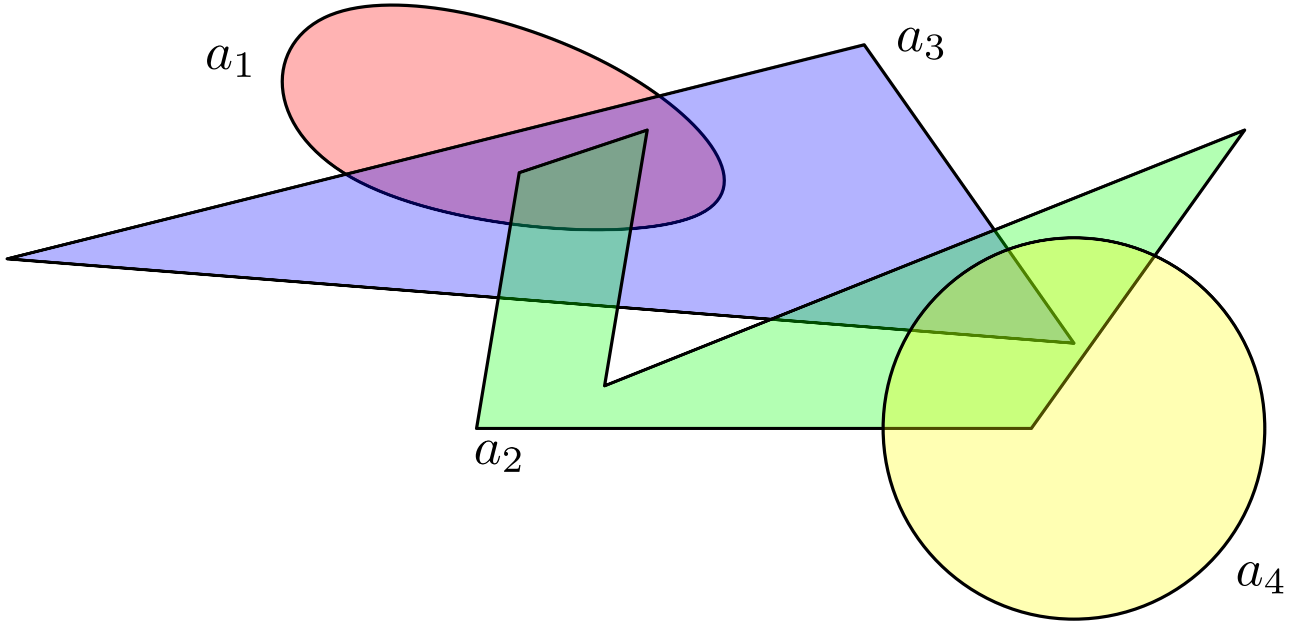

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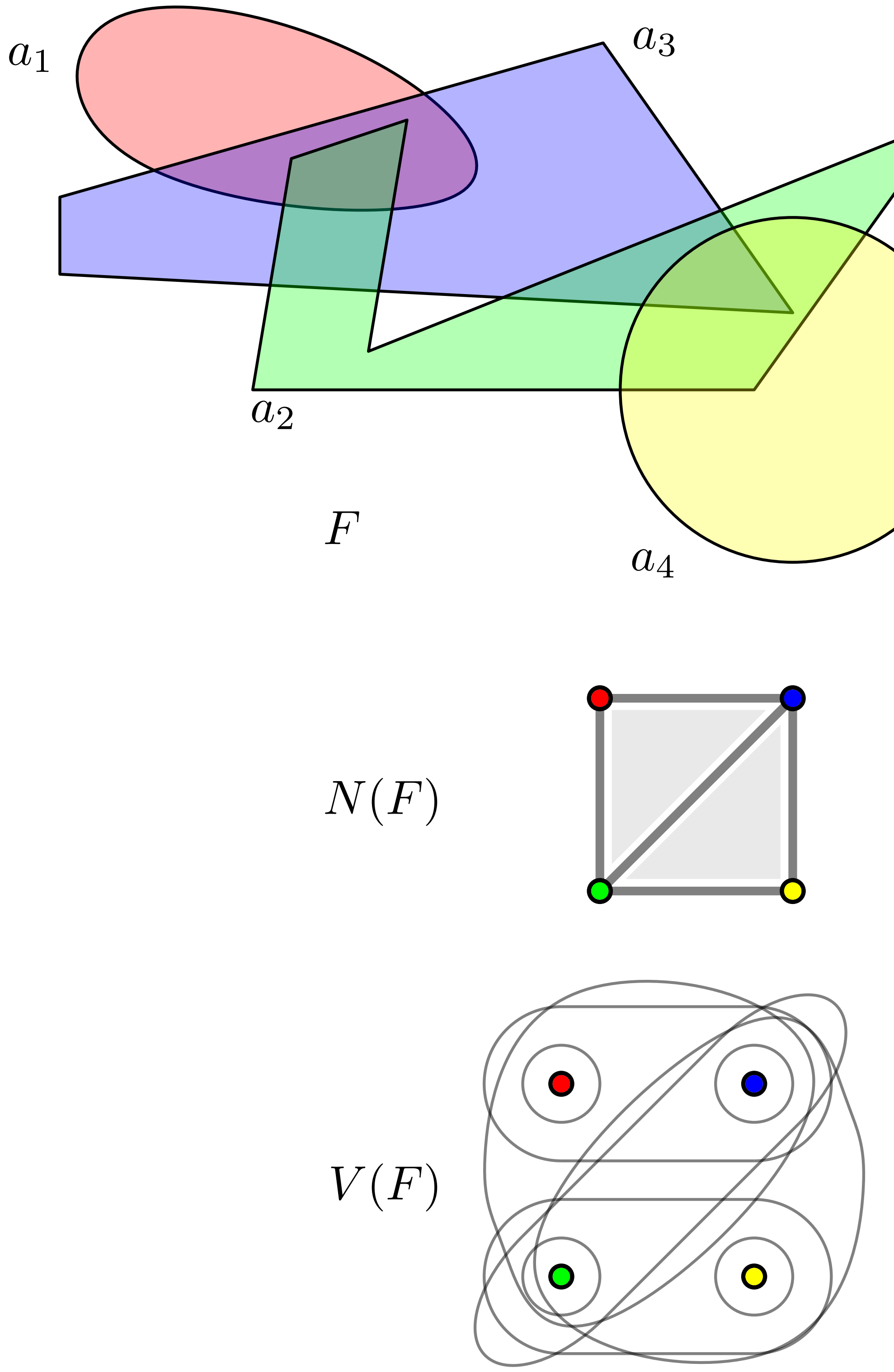
- ▷ ... exponentially many variables, some NP-hard variant.

#2. A ”topological” shortcut

Combinatorics, Probability and Computing: page 1 of 19. © Cambridge University Press 2014
doi:10.1017/S096354831400042X

Simplifying Inclusion–Exclusion Formulas

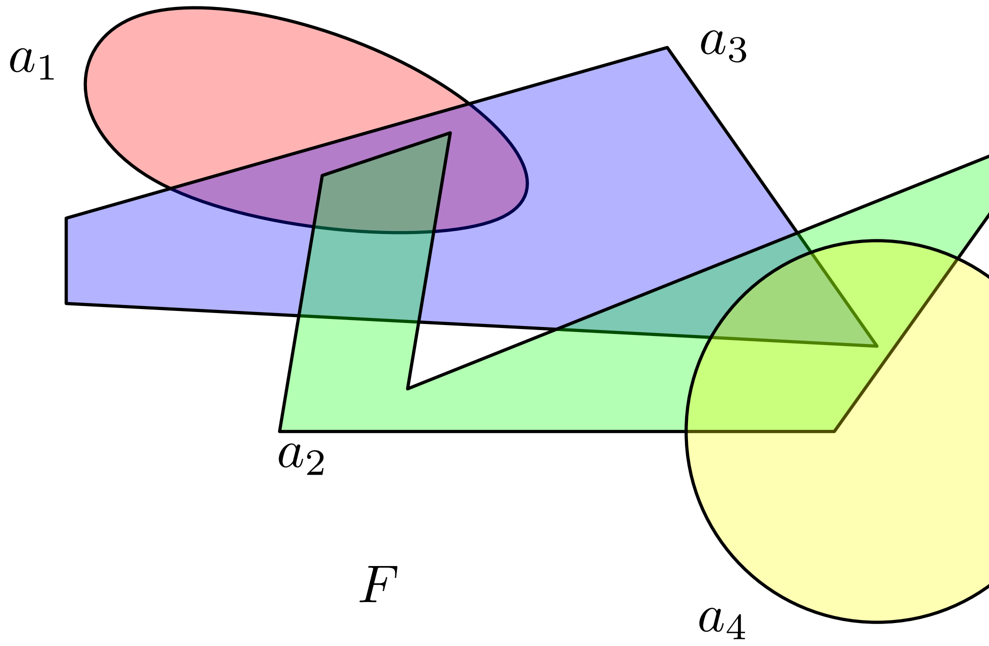
$K \subseteq 2^{[n]}$ an (abstract) simplicial complex.



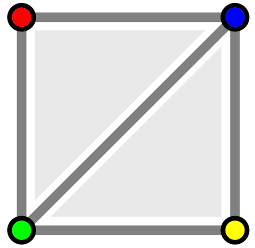
$K \subseteq 2^{[n]}$ an (abstract) simplicial complex.

- $$\mathbb{1}_{\bigcup_{i=1}^n a_i} = \sum_{\sigma \in K} (-1)^{\dim \sigma} \mathbb{1}_{\bigcap_{i \in \sigma} a_i}$$

  $$\Leftrightarrow \quad \forall \tau \in V(F),\ \chi(K[\tau]) = 1.$$



$N(F)$



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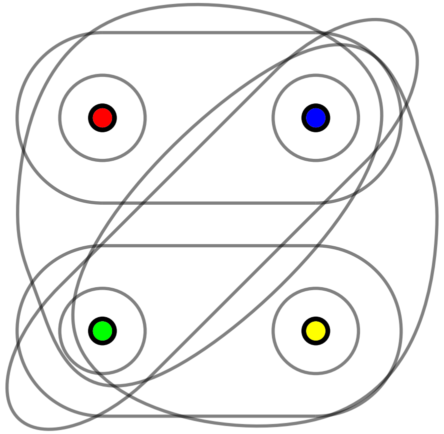

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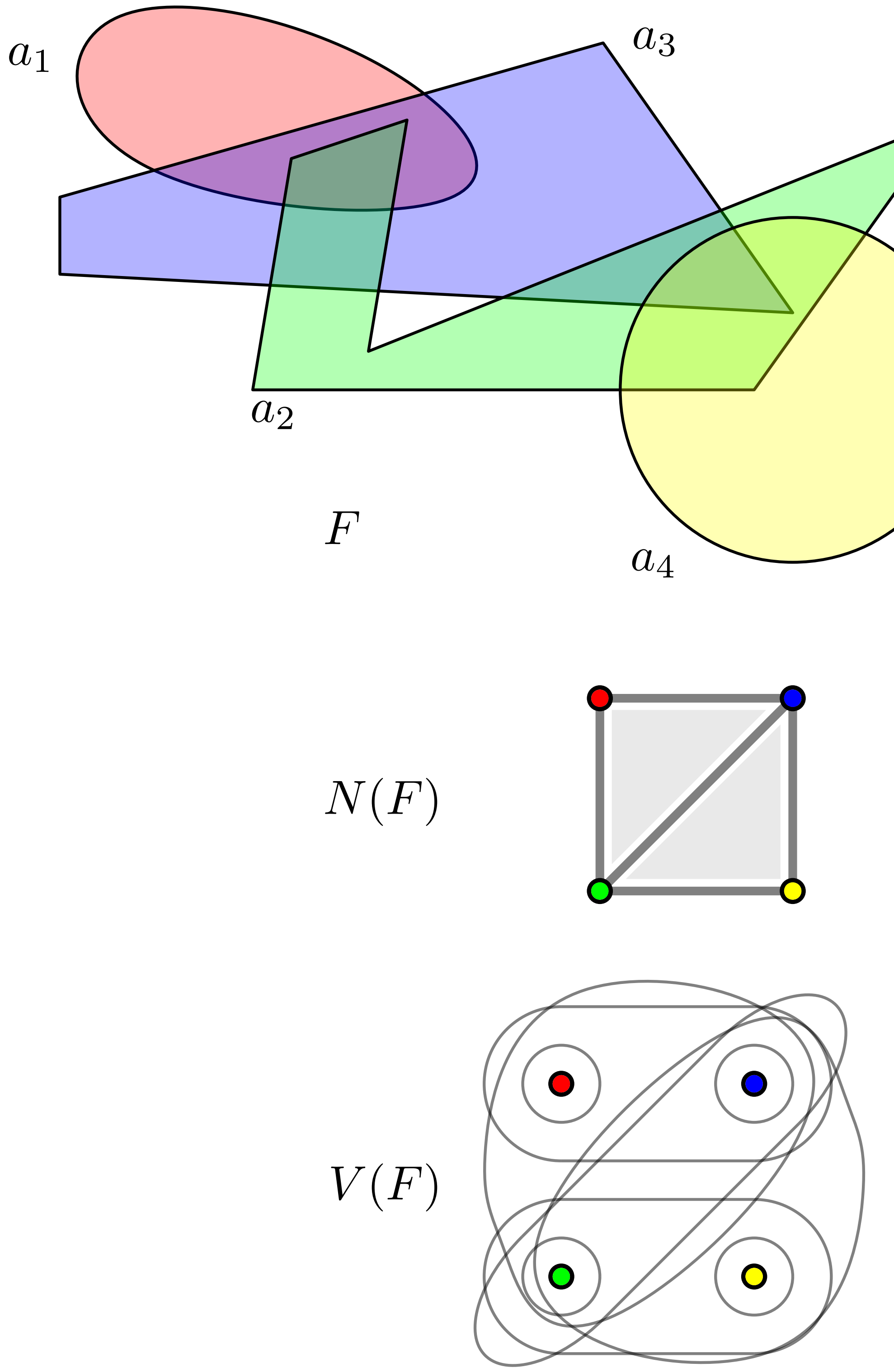

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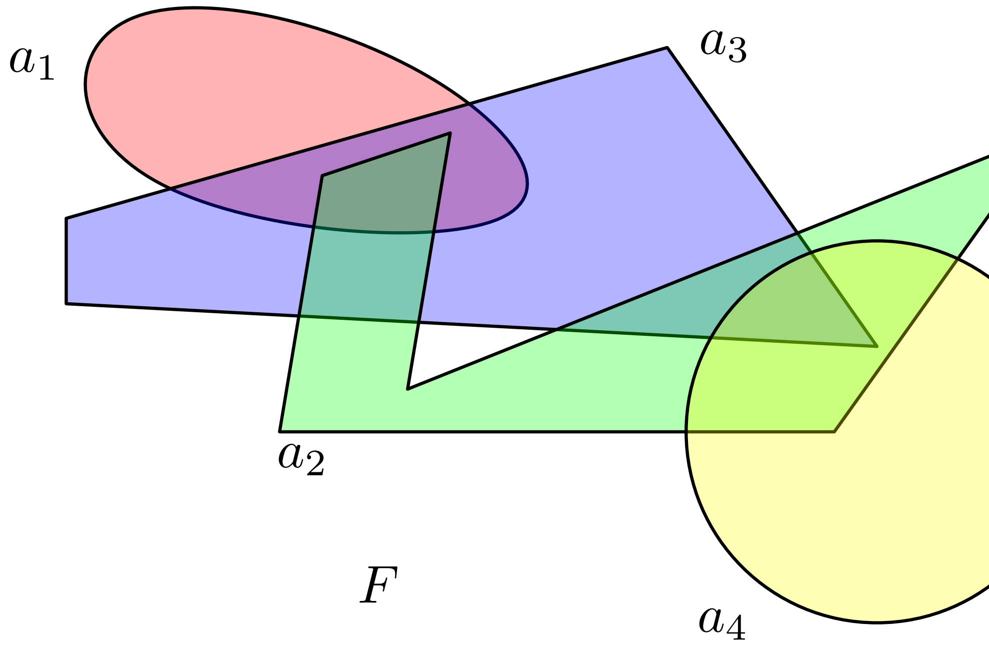

An abstract simplicial complex C is a **cone**
if there exists a vertex $\{v\} \in C$ s.t. $\forall \sigma \in C$, $\sigma \cup \{v\} \in C$.

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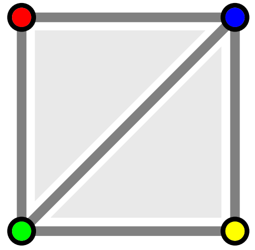

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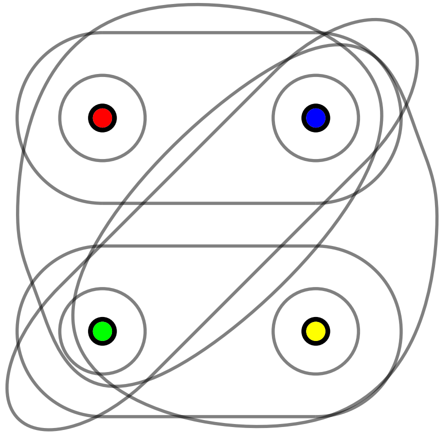

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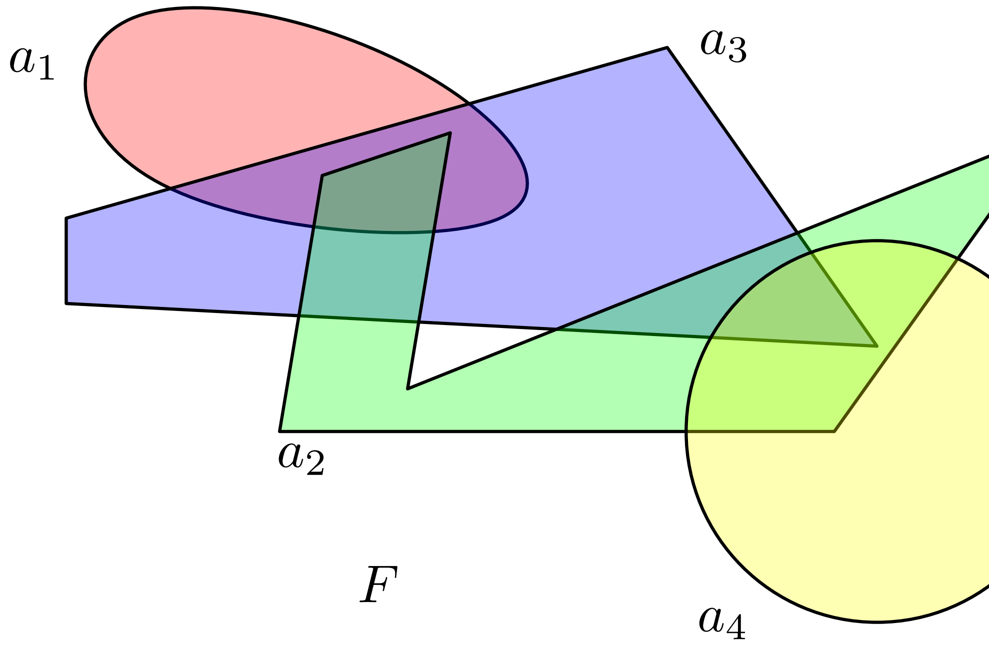

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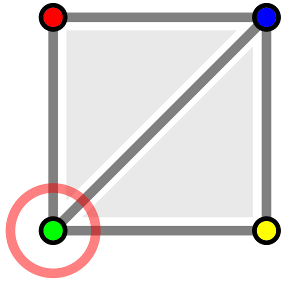

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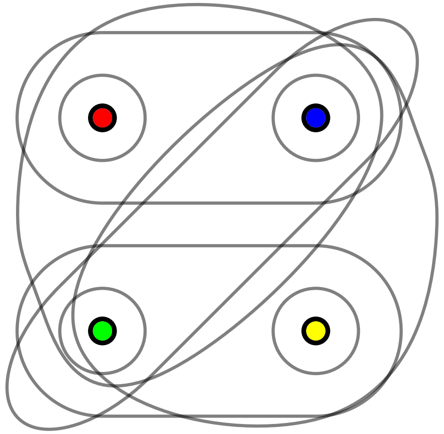

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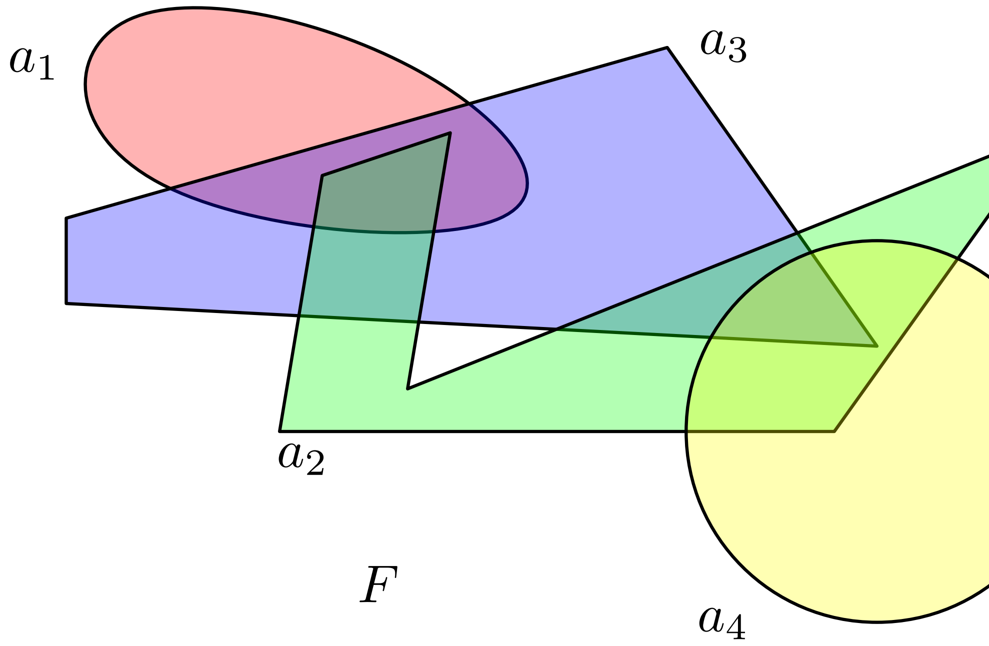

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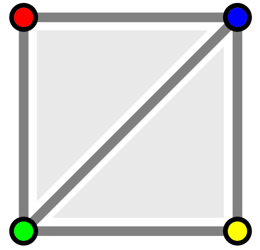

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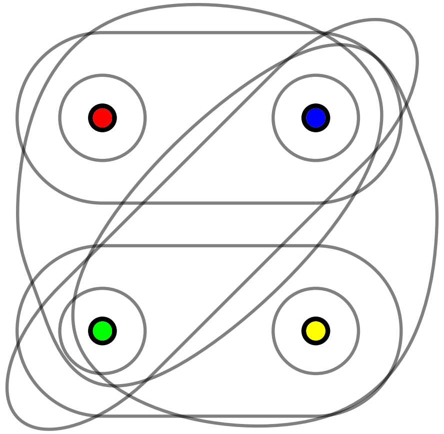
- $K[\bullet]$ a cone $\Rightarrow \chi(K[\bullet]) = 1$

  pairing-up, $|$cone$|$ is contractible, ...

$V(F)$



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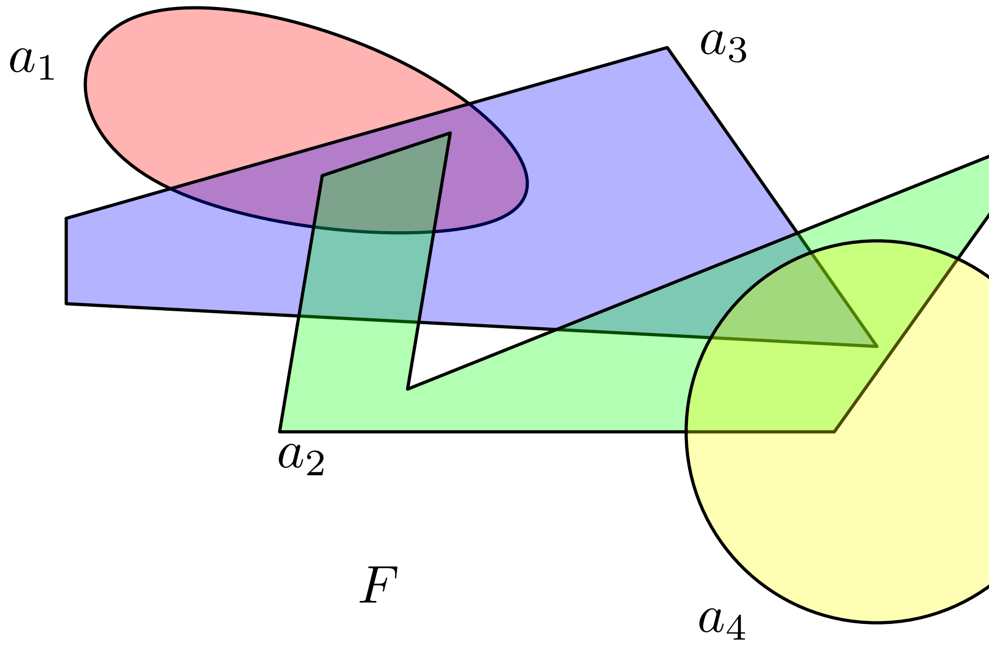

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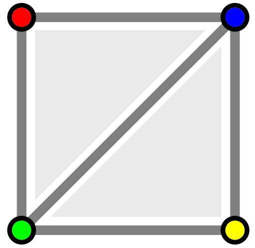

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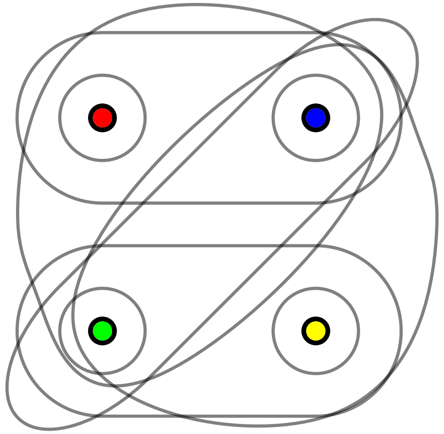

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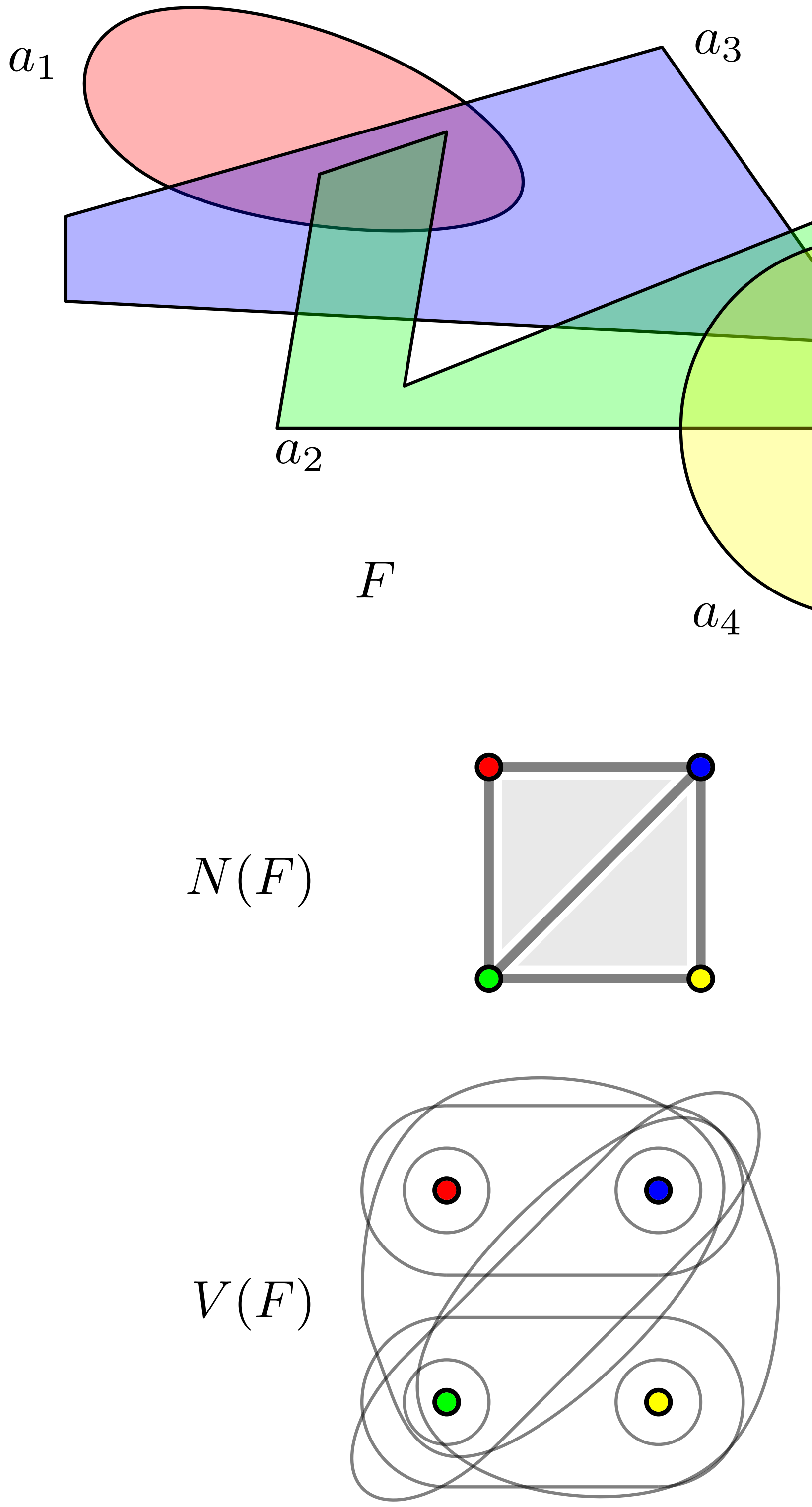

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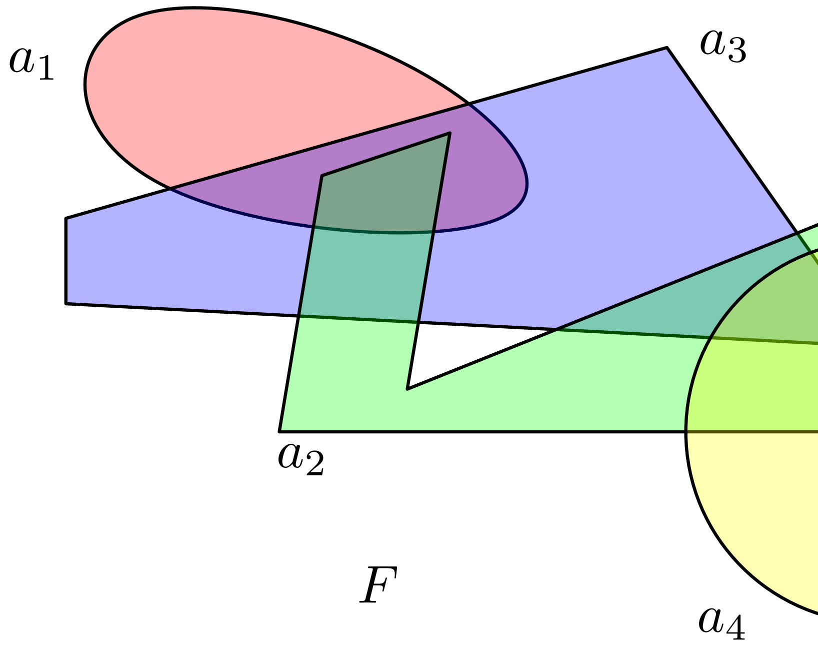
Lemma. If no element of $V(F)$ is a union of inclusion-minimal non-faces of K, then (*) holds.



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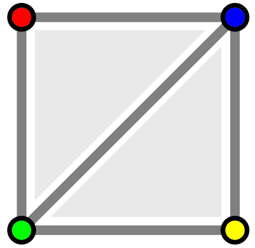

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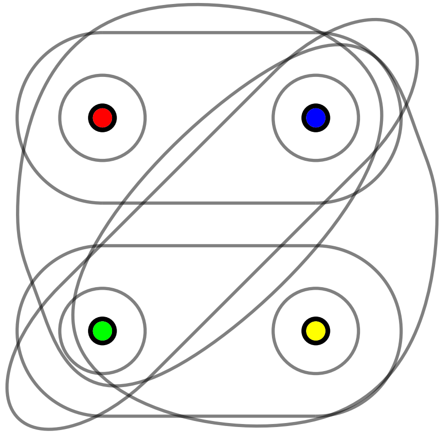

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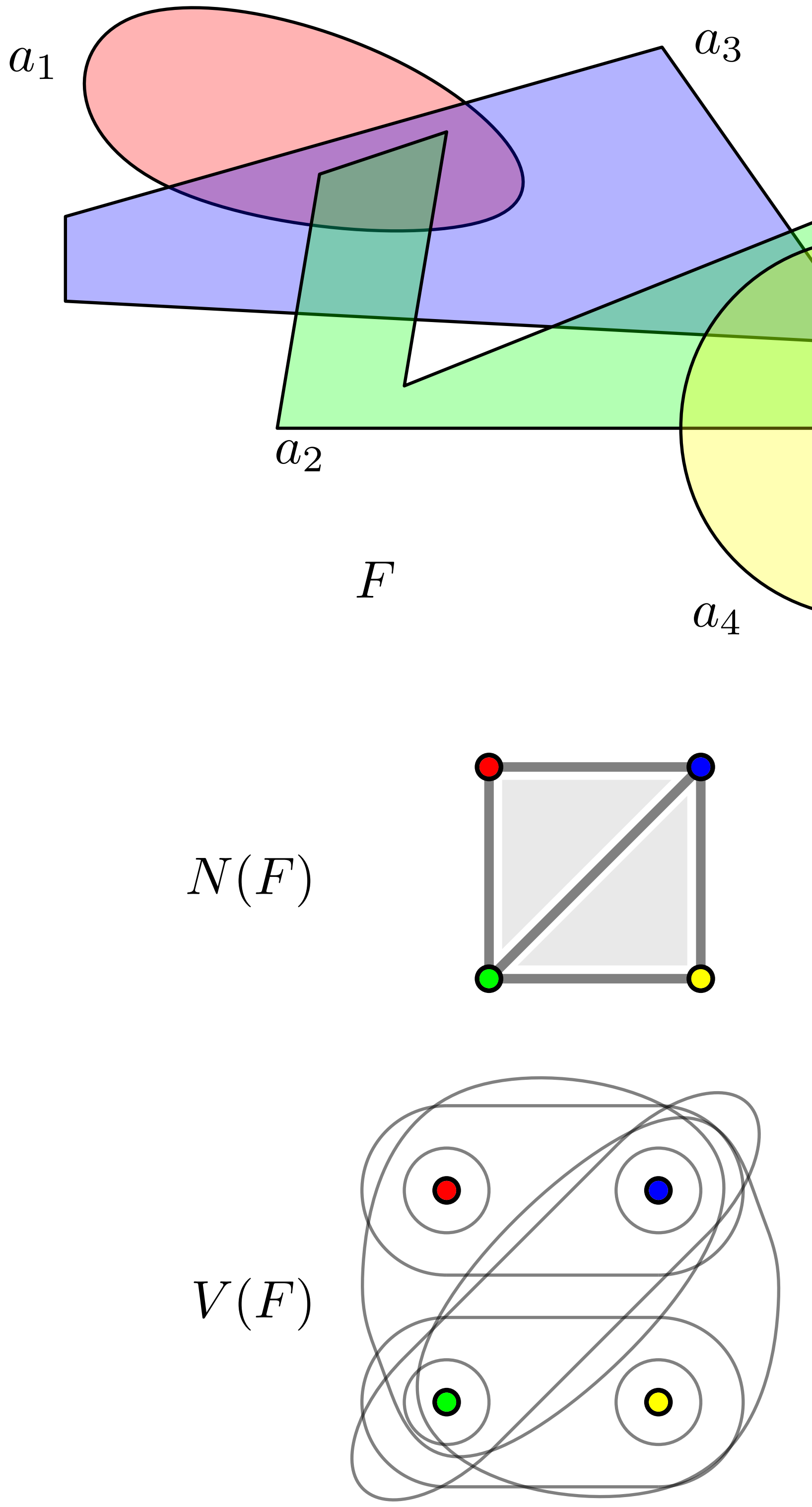

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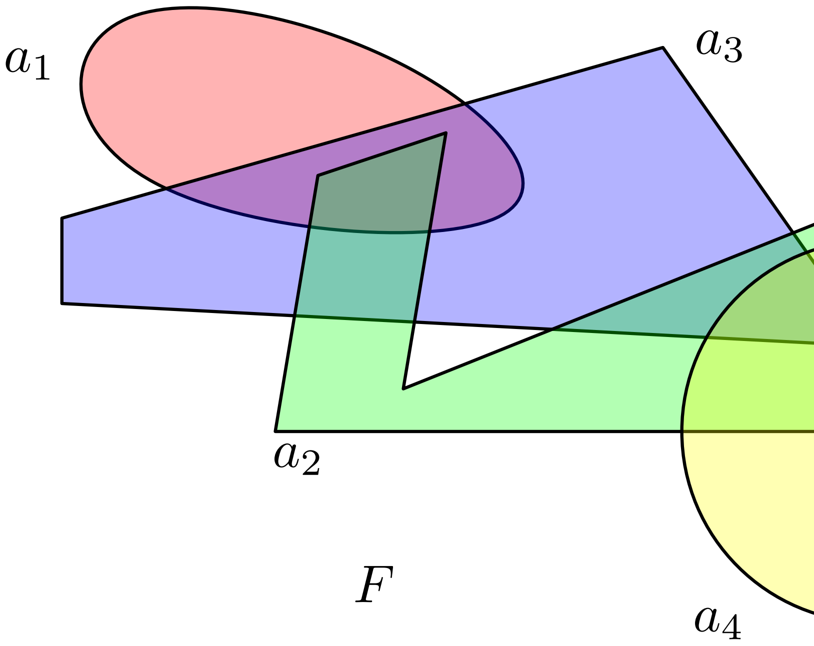

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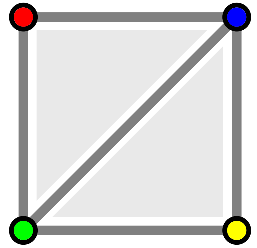

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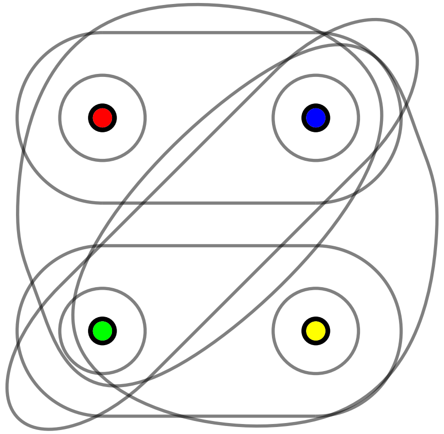

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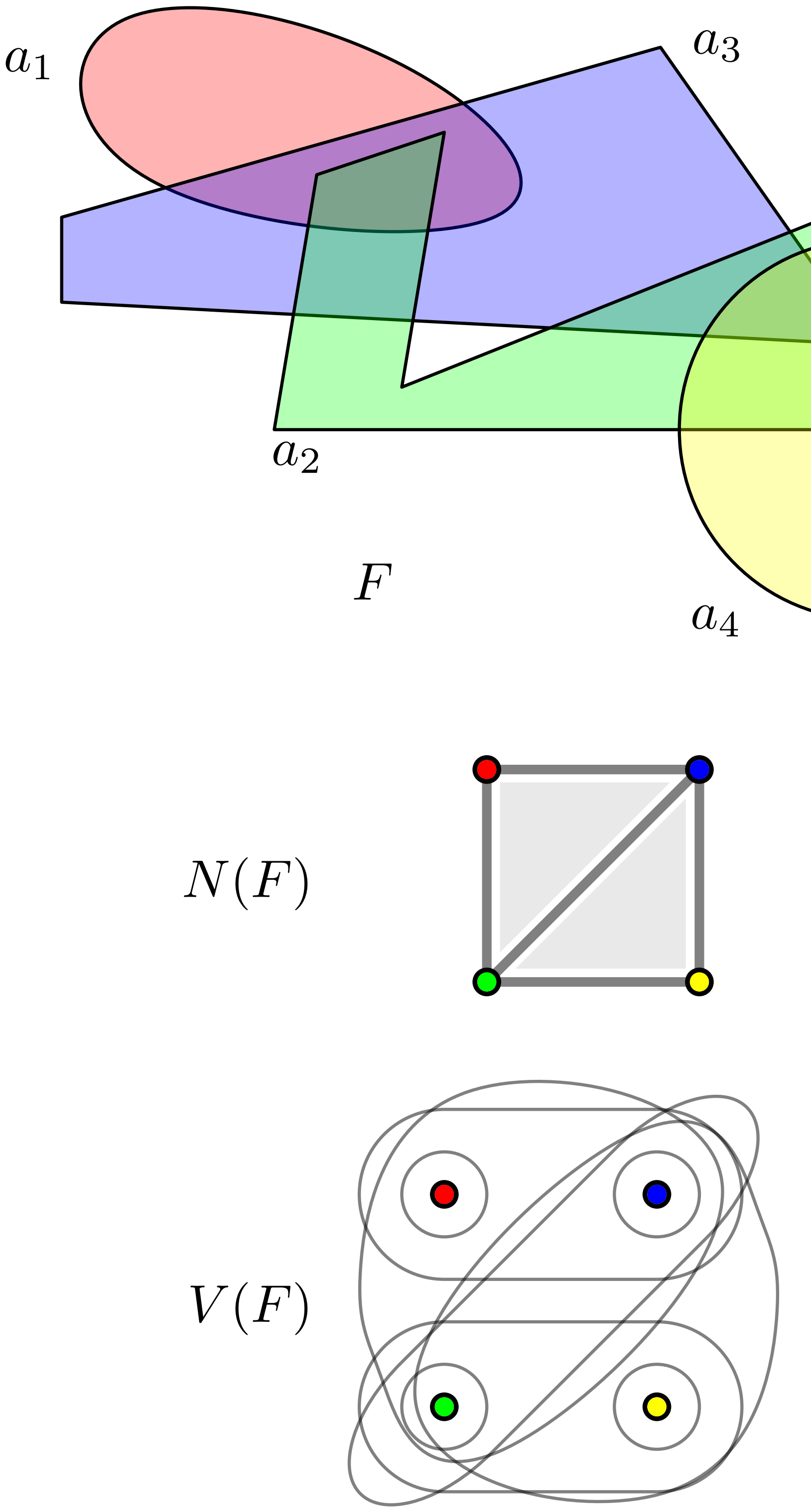

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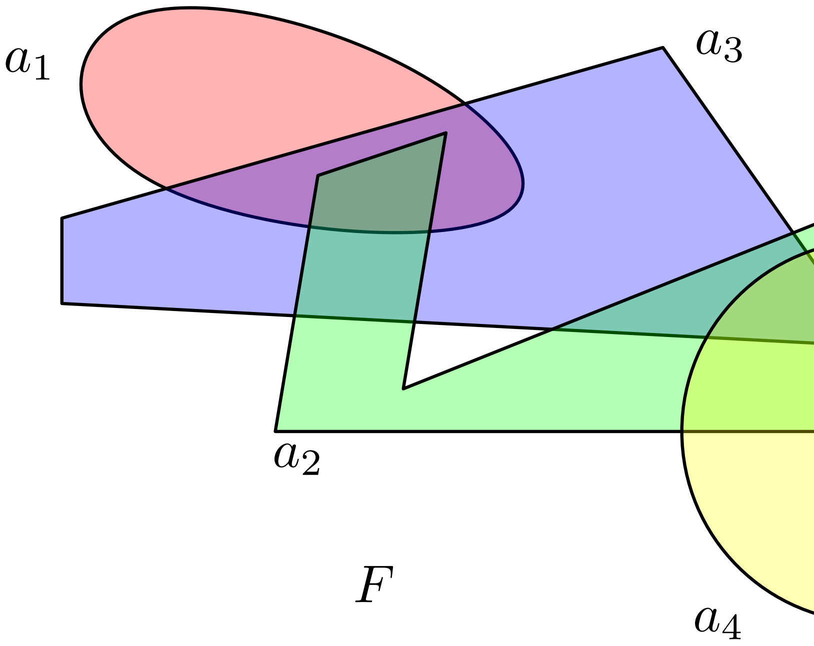
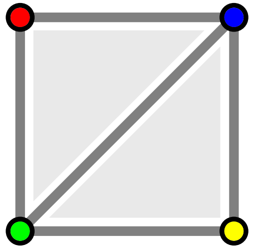


F

Lemma. If no element of $V(F)$ is a union of inclusion-minimal non-faces of K, then (*) holds.

$N(F)$



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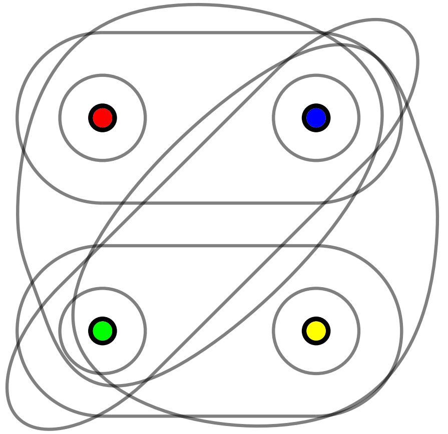

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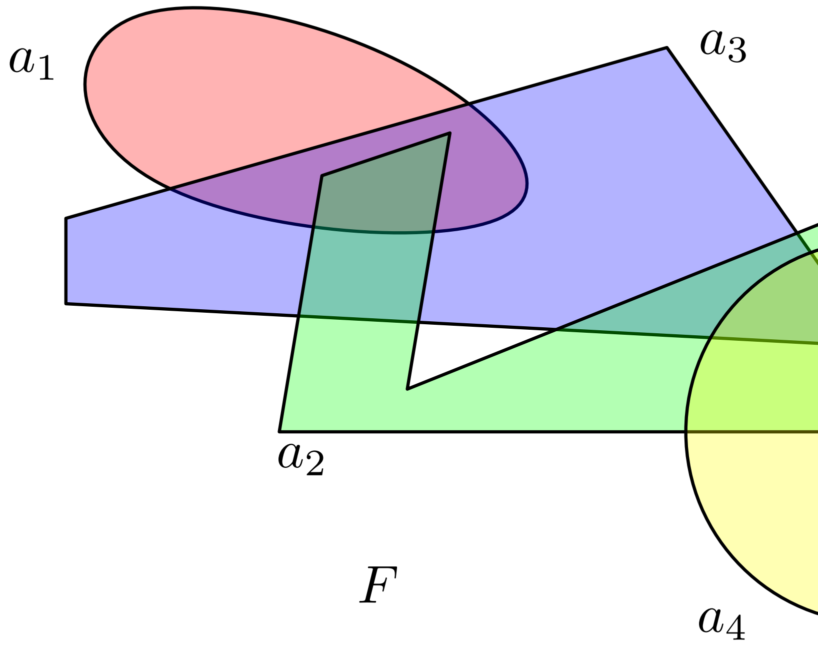

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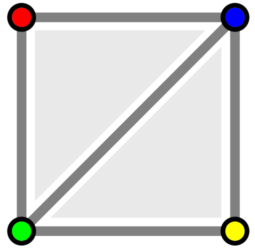

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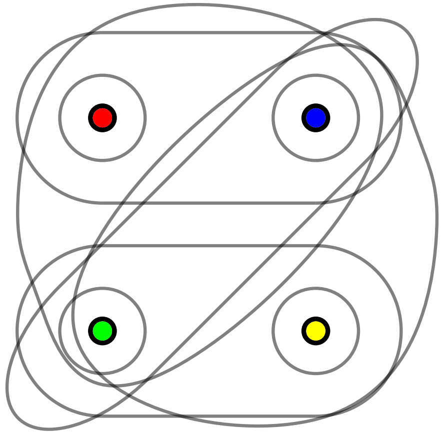

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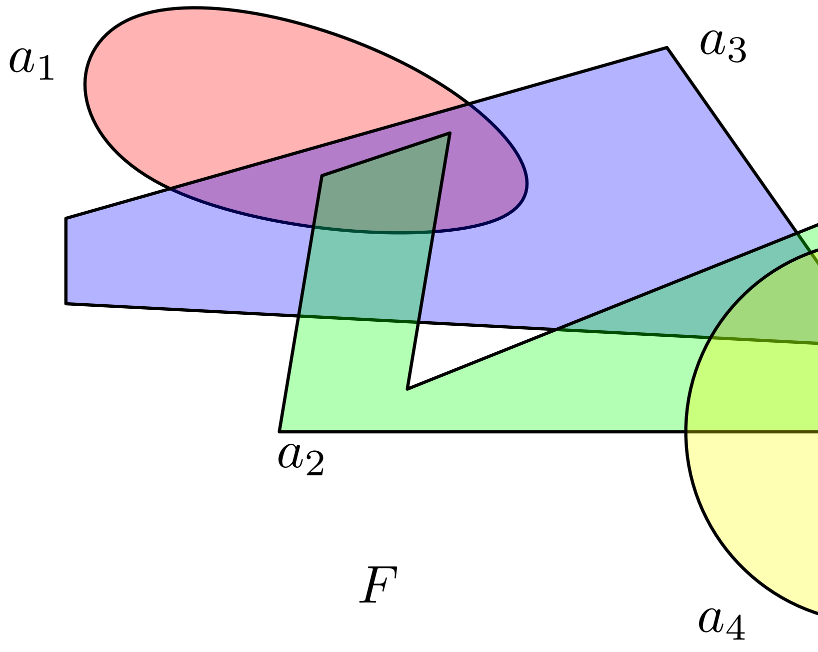

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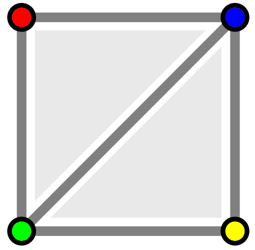

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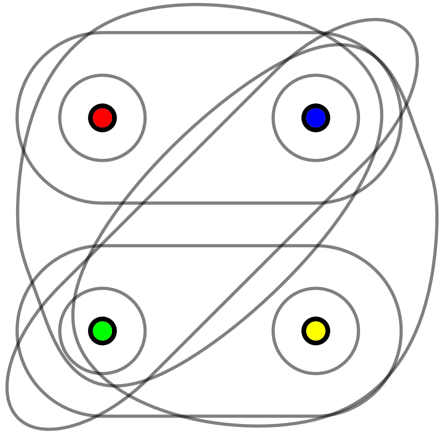

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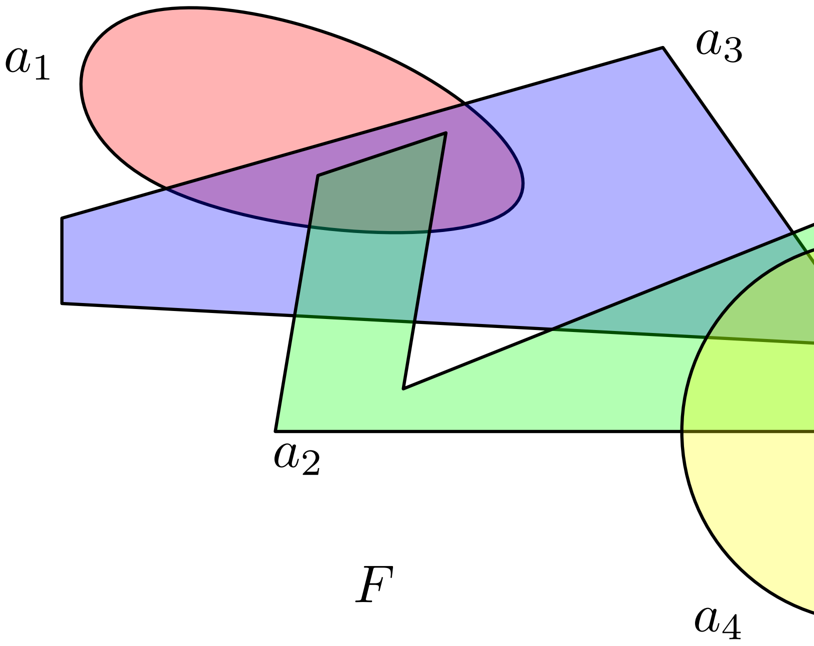

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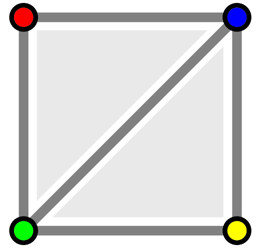

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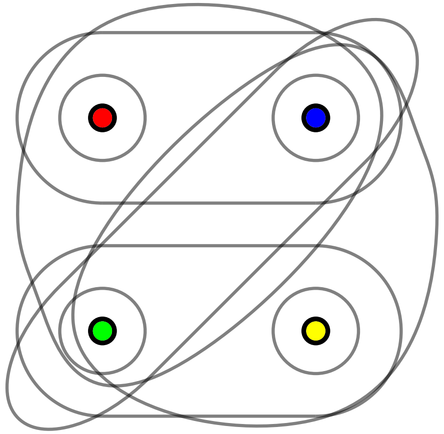

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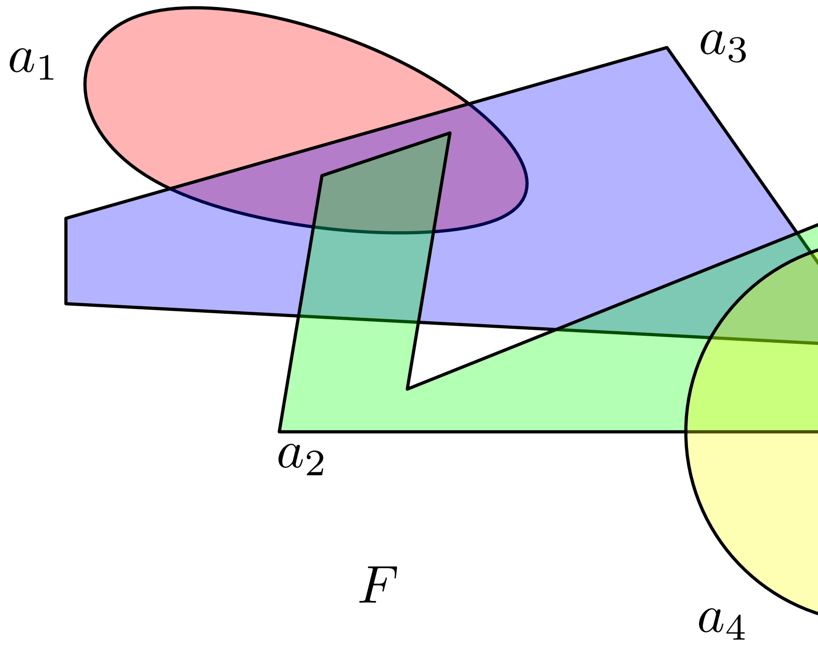

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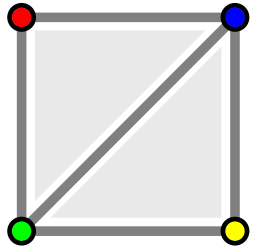

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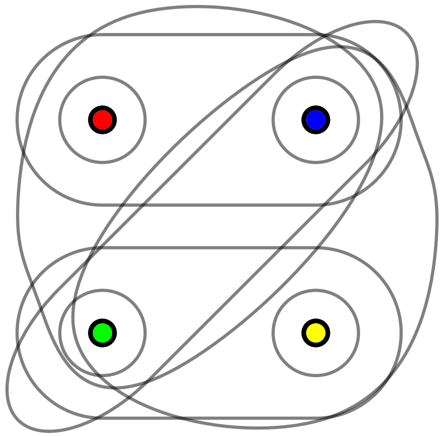

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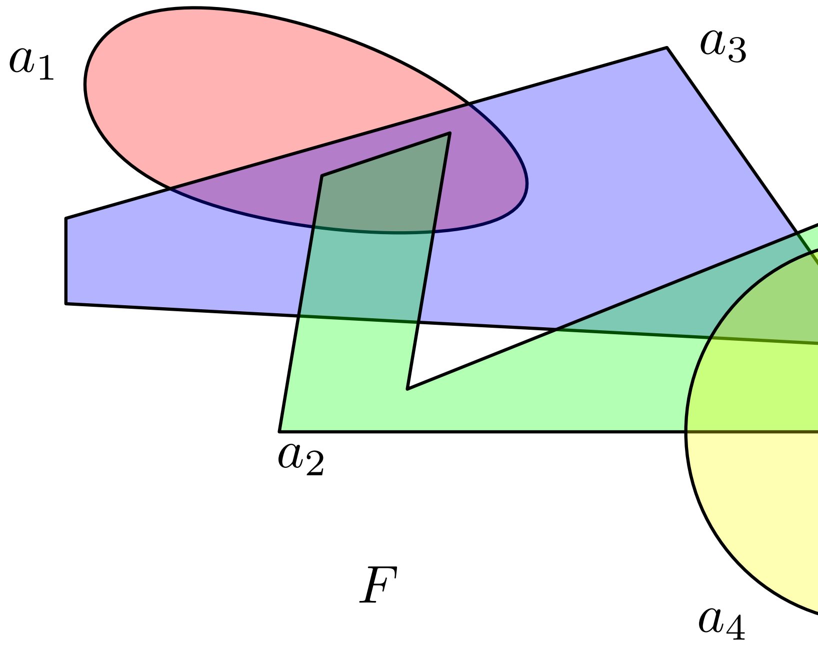

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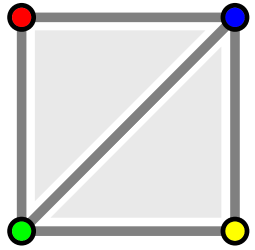

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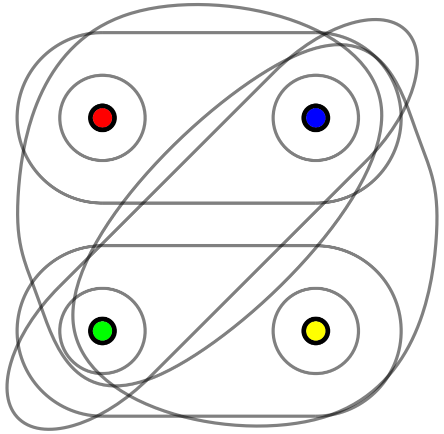

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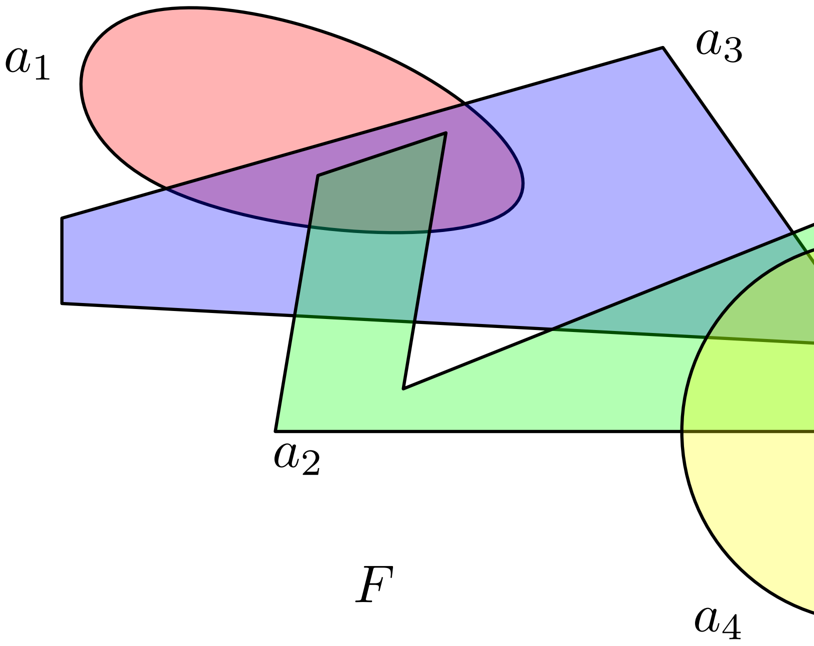

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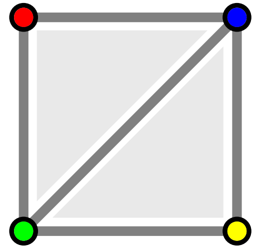

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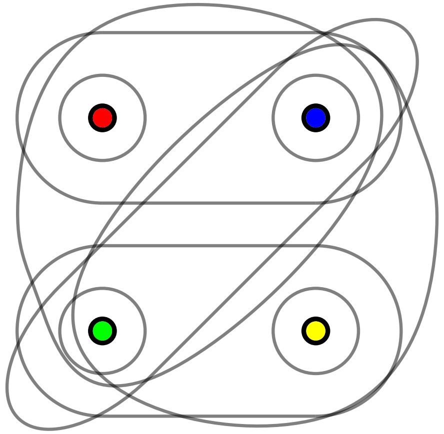

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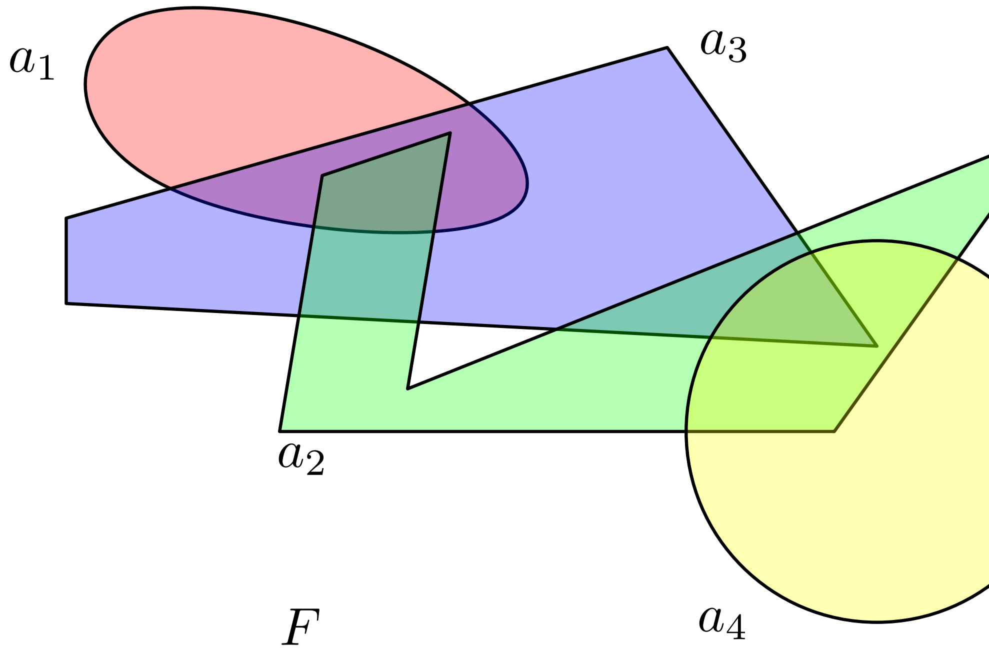

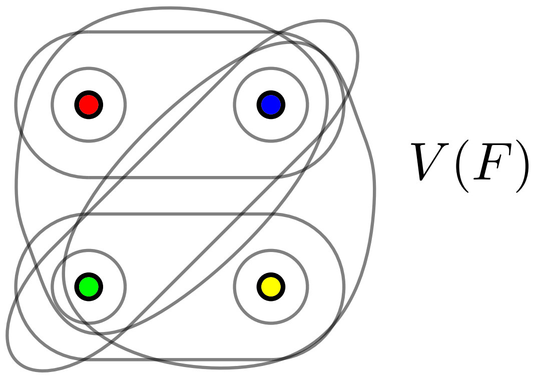

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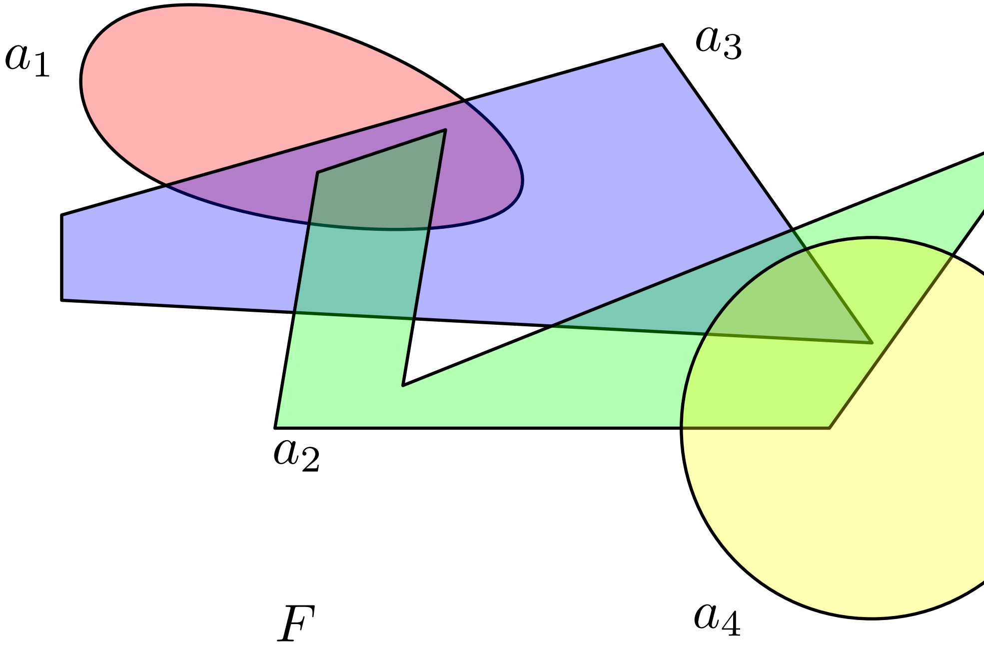

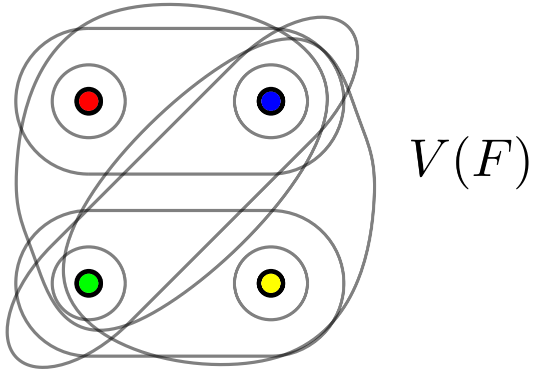

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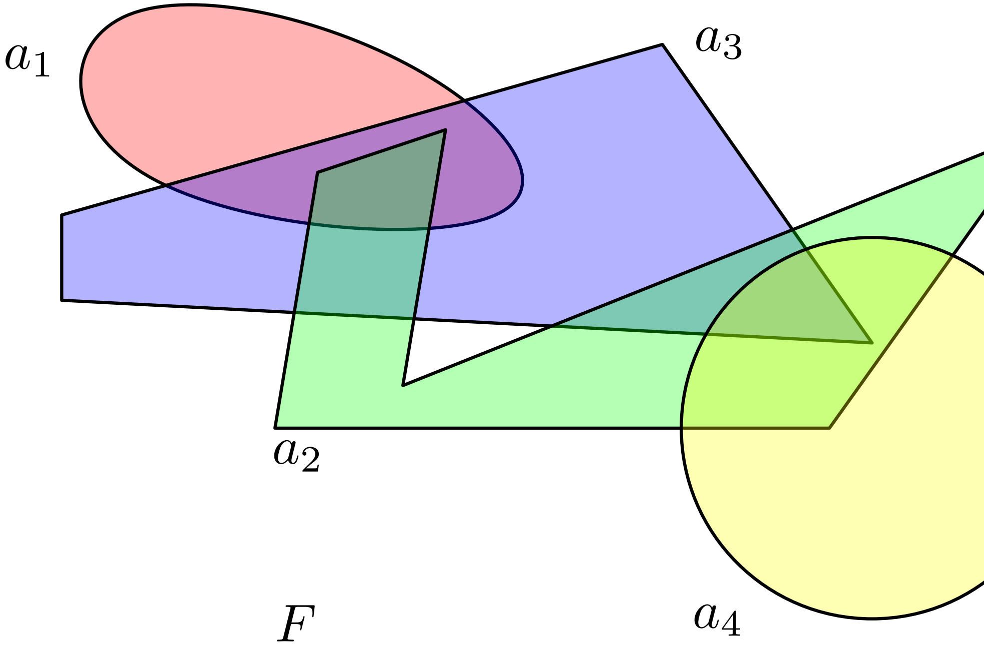

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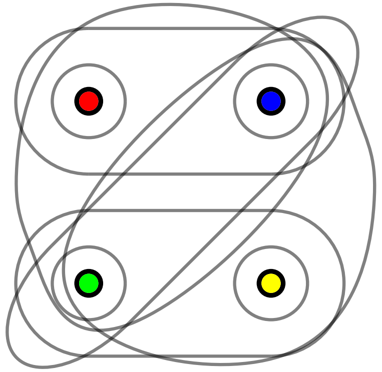

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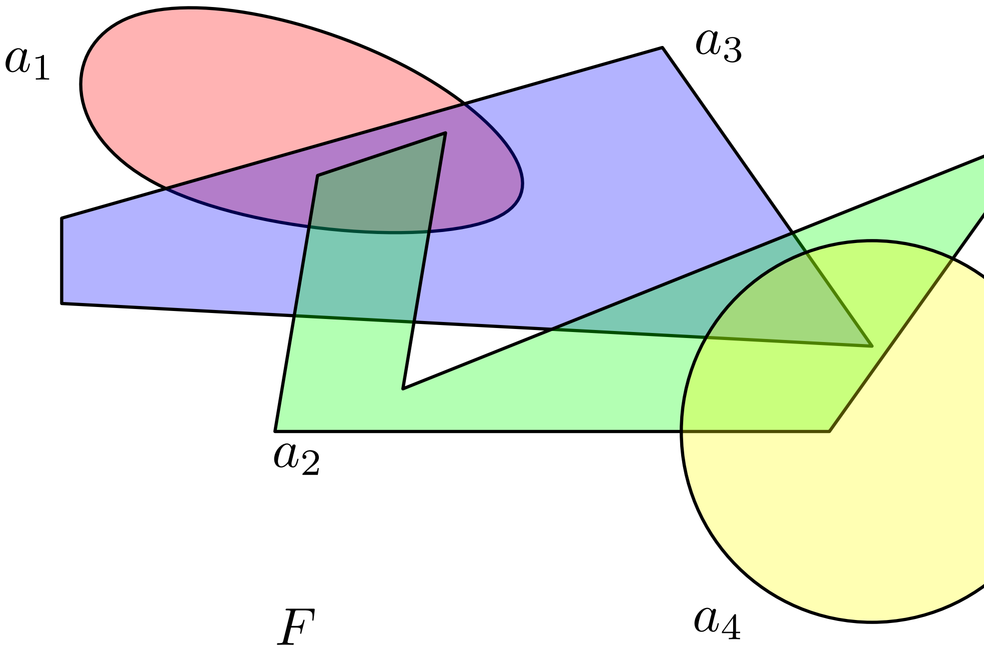

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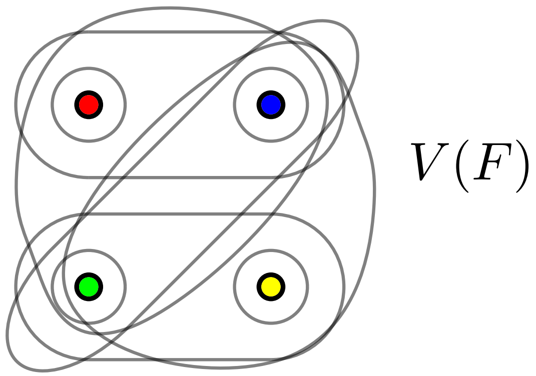

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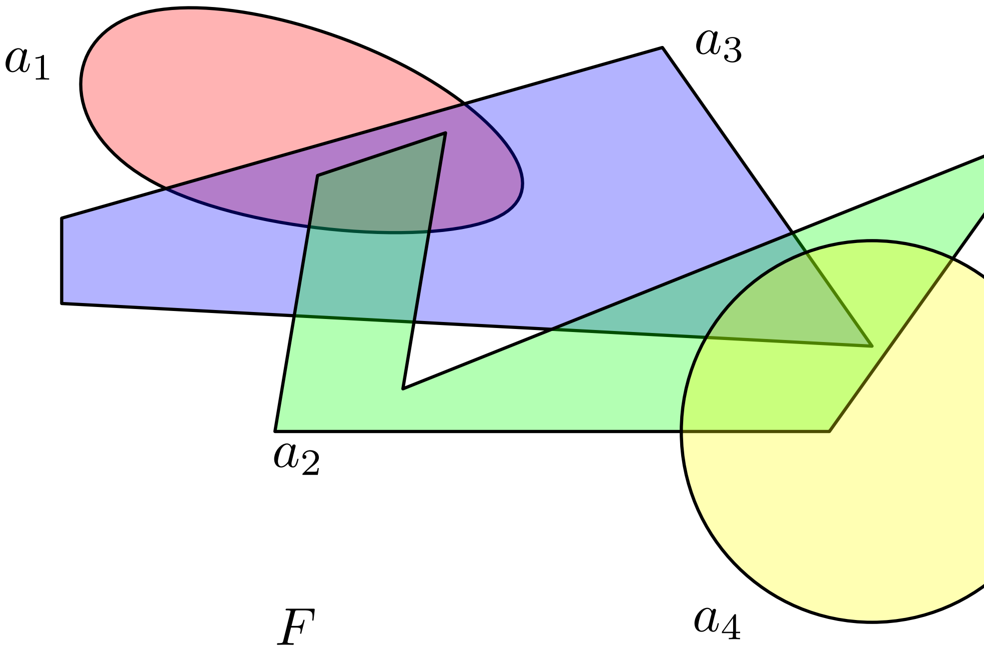

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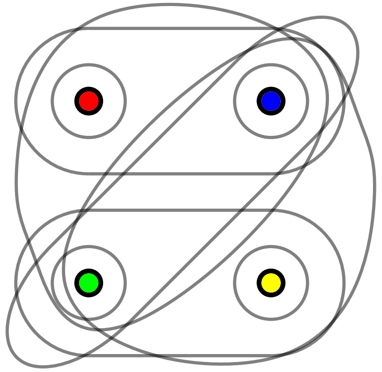

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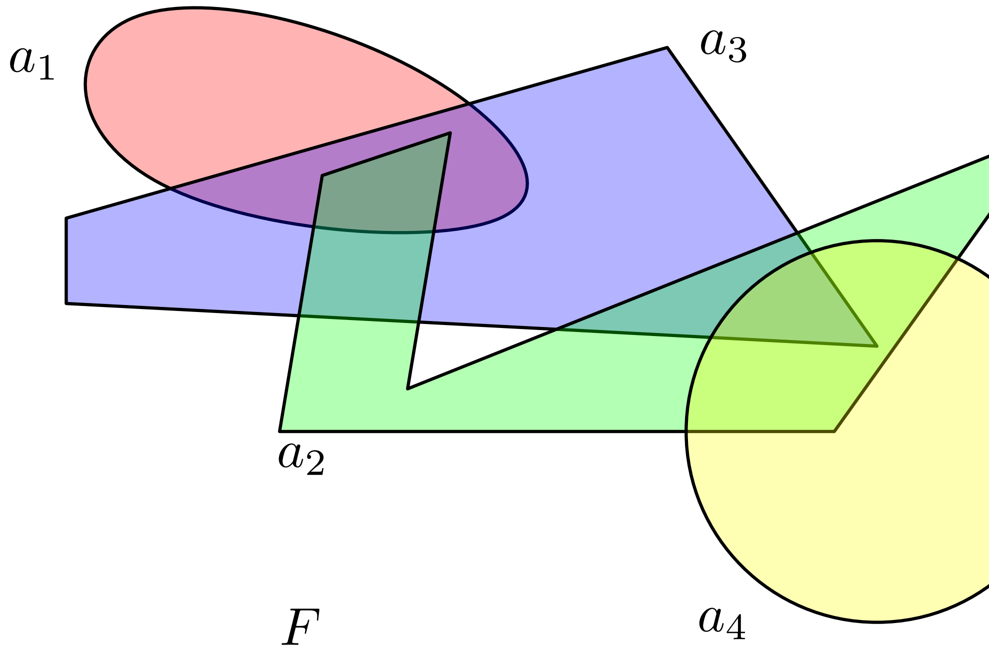

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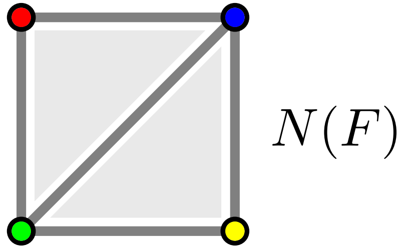

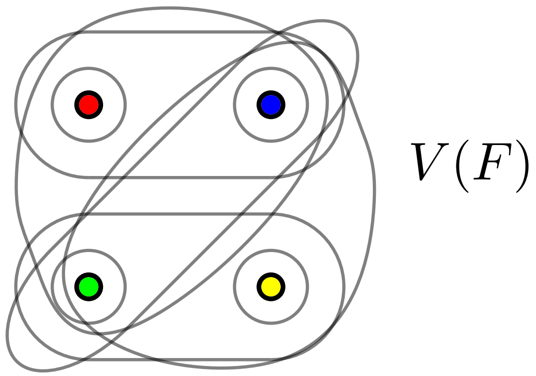

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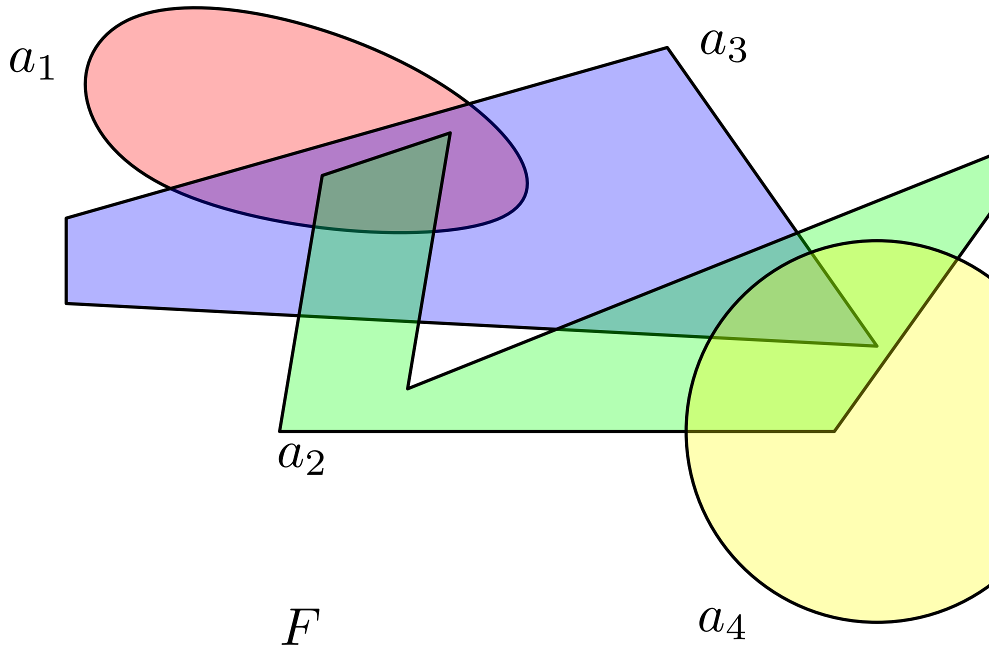

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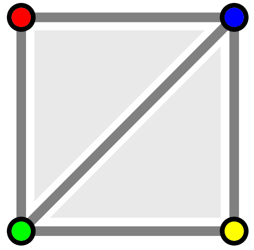

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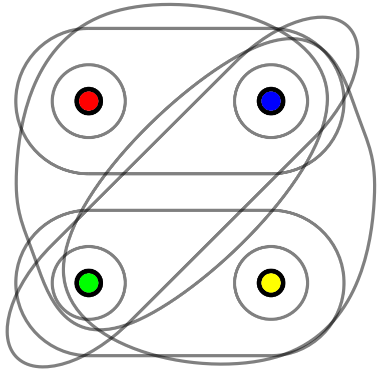

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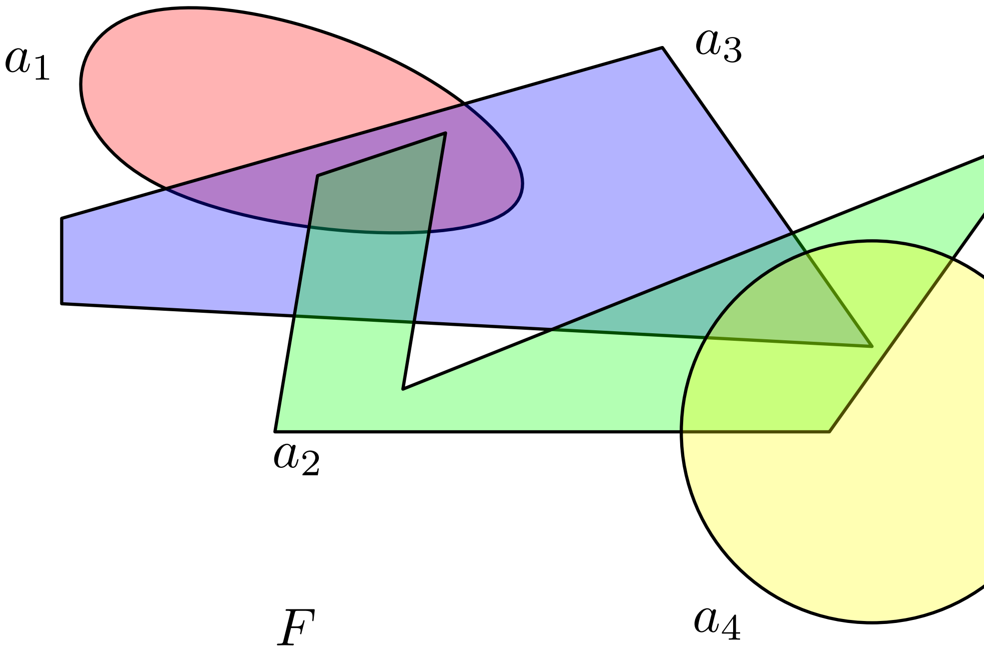

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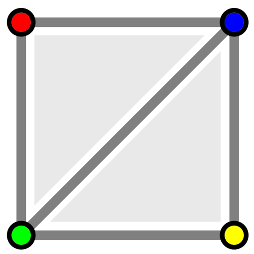

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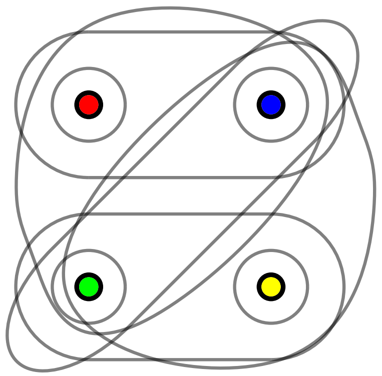

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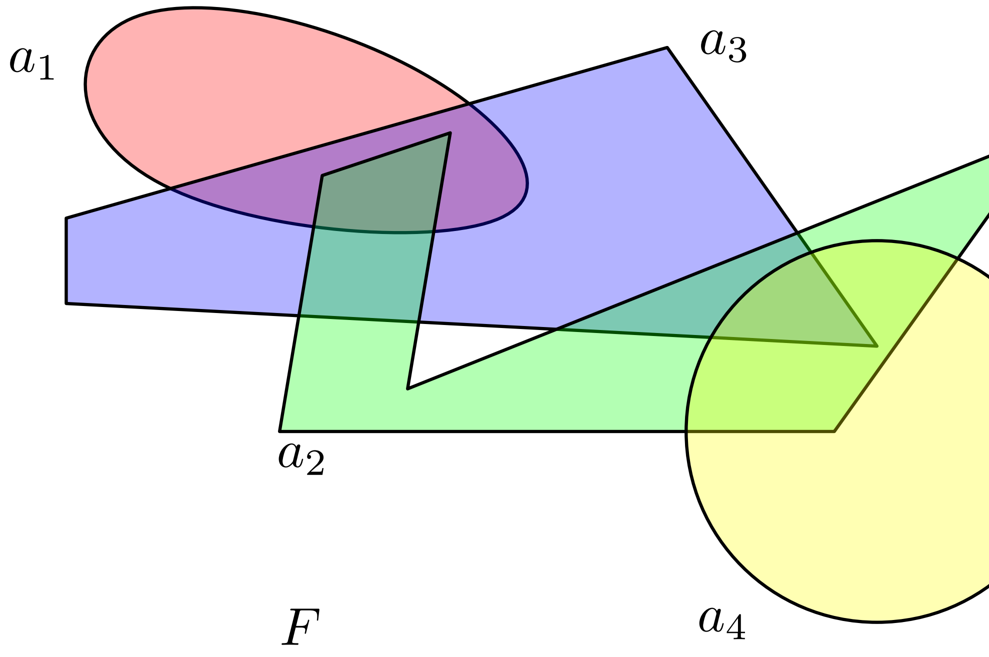

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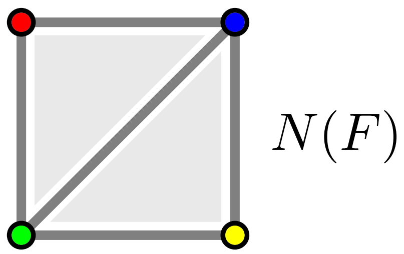
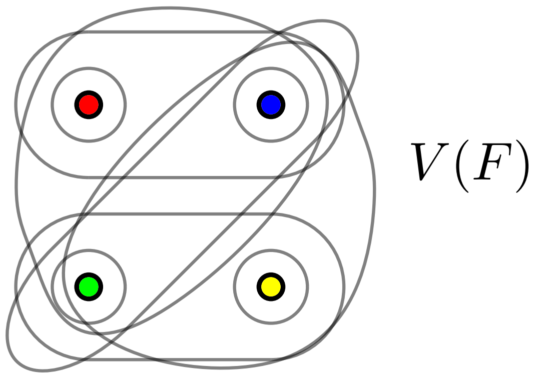
Def. $\Rightarrow$ for every $\tau \in V(F)$, $w(\tau)$ is not in min. non-faces contained in τ.

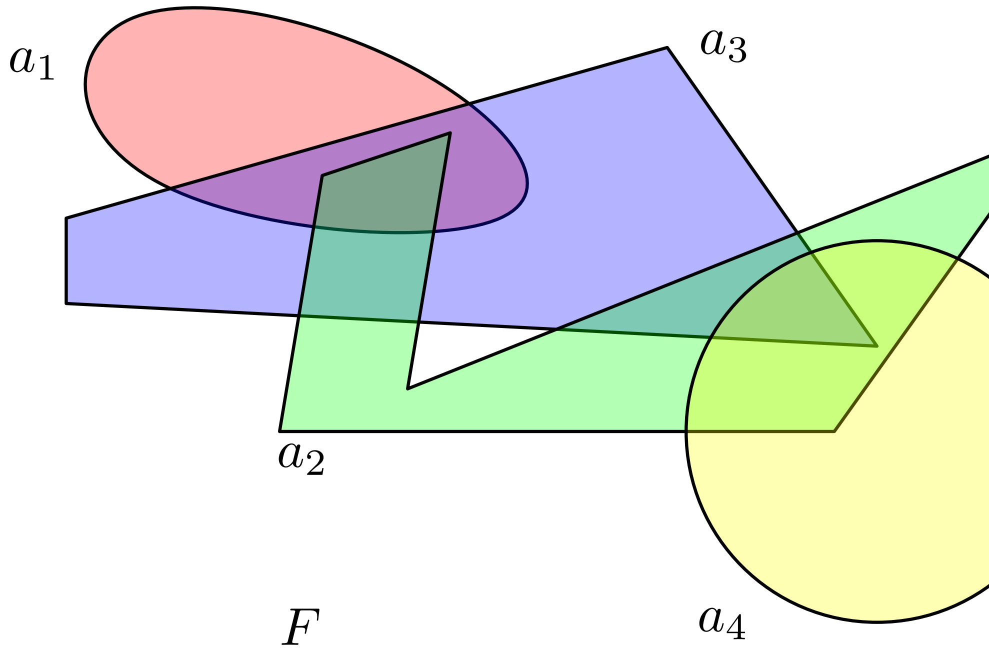




$F = \{a_1, a_2, \ldots, a_n\}$ a set system, $V(F)$ its Venn diagram.

Goal: $K \subseteq 2^{[n]}$ s.t. $\forall \tau \in V(F)$, $K[\tau]$ is a cone.

$\Leftrightarrow$ *no $\tau \in V(F)$ is a union of min. non-faces of K.*



Algorithm:

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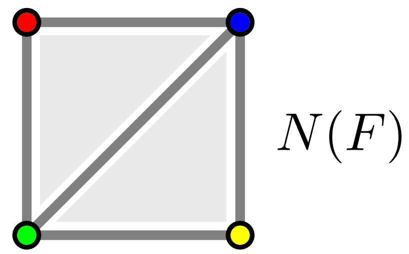
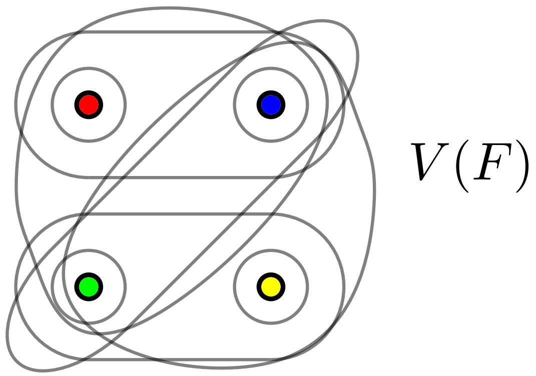
- ▷ Define $K_{w(\cdot)} = \{\sigma \in N(F) \colon \forall \theta \subseteq \sigma, \exists \tau \in V(F) \text{ s.t. } w(\tau) \in \theta \subseteq \tau\}$.
- ▷ For every selector $w(\cdot)$, $\forall \tau \in V(F)$, $K_{w(\cdot)}[\tau]$ is a cone.
- ▷ Given $w(\cdot)$, K can be constructed bottom-up.





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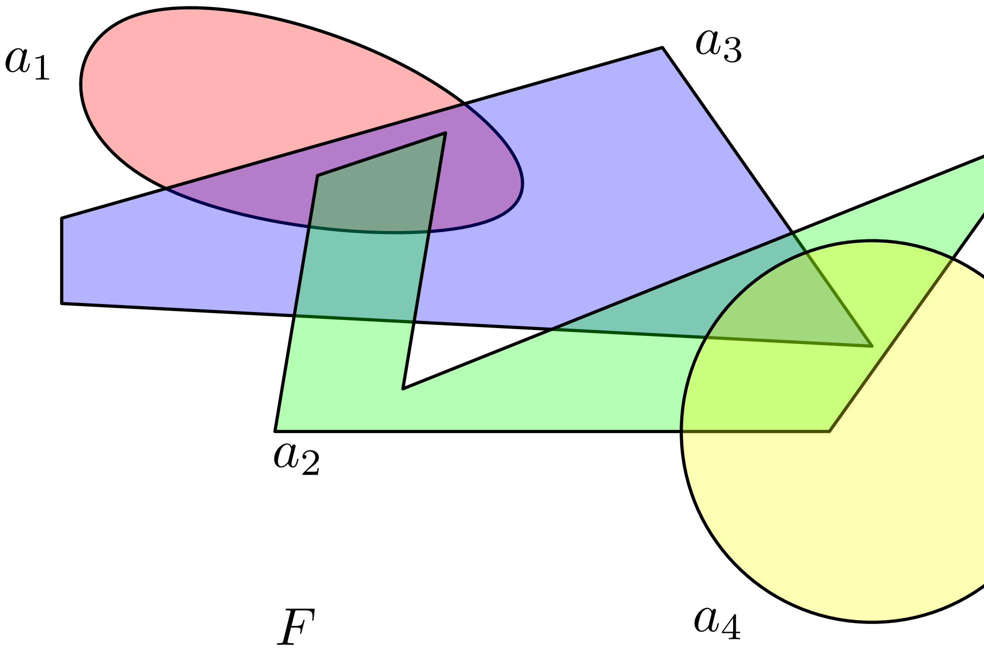

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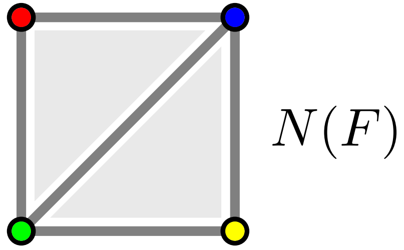

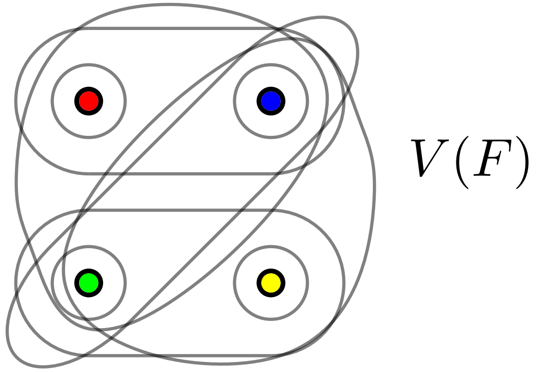

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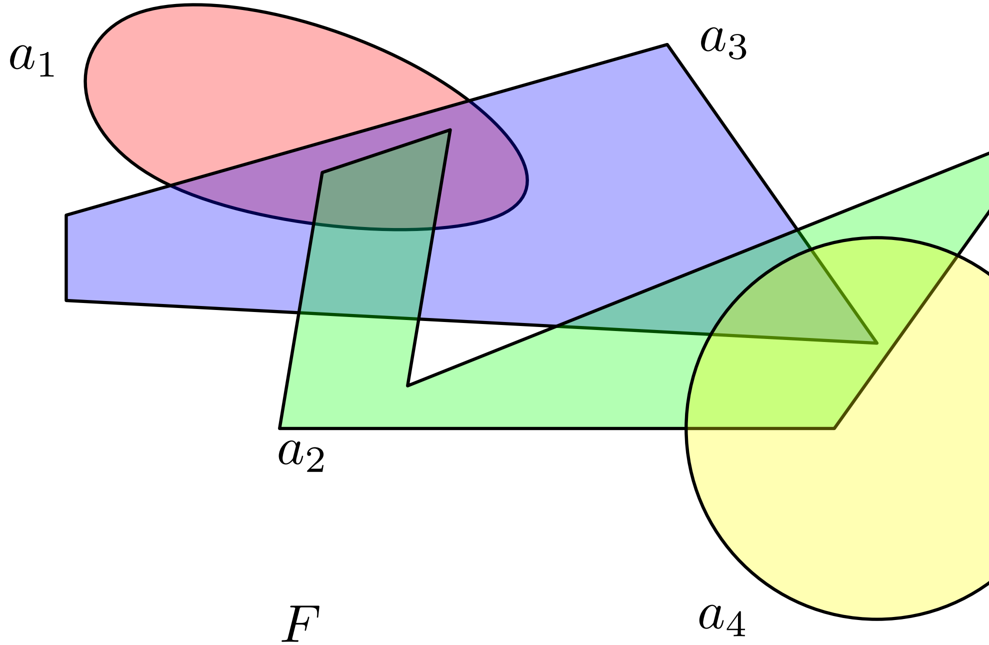

F

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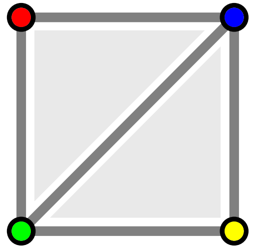

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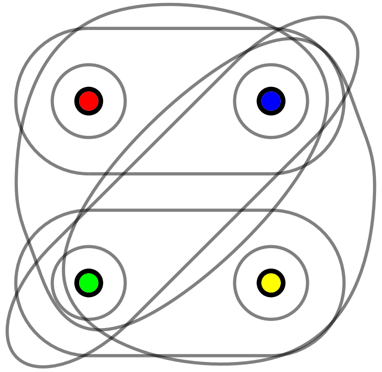

$V(F)$

Is there a selector $w(\cdot)$ such that $K_{w(\cdot)}$ is small?

●● ●●● ○○ ○ ○ ○

#3. An ad hoc model of random simplicial complexes

Combinatorics, Probability and Computing: page 1 of 19.
doi:10.1017/S096354831400042X

Simplifying Inclusion–Exclusion Formulas

Fix a **permutation** ρ on $[n]$,
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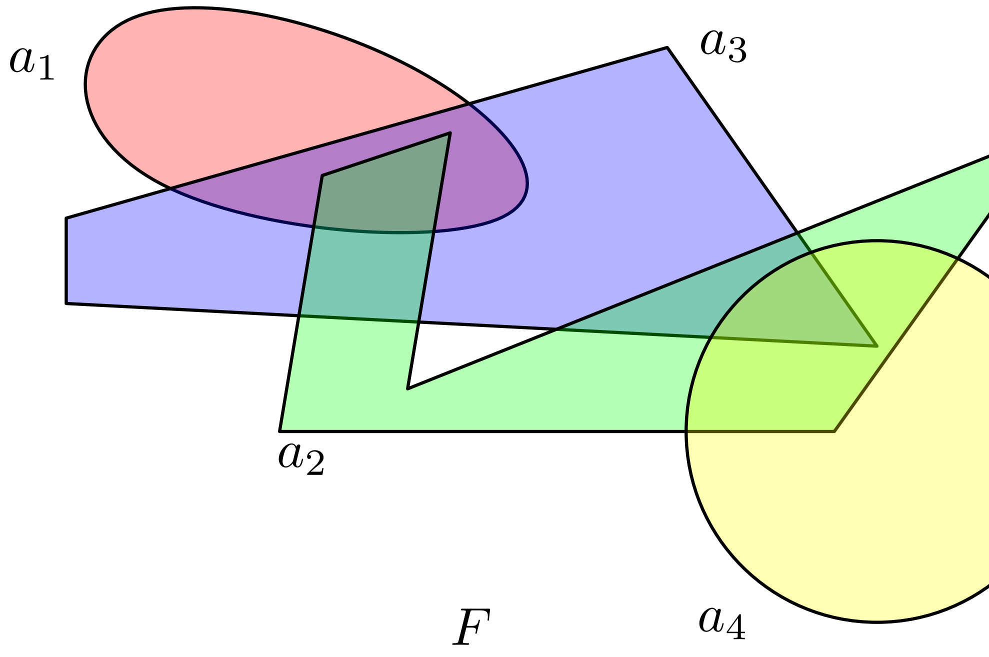

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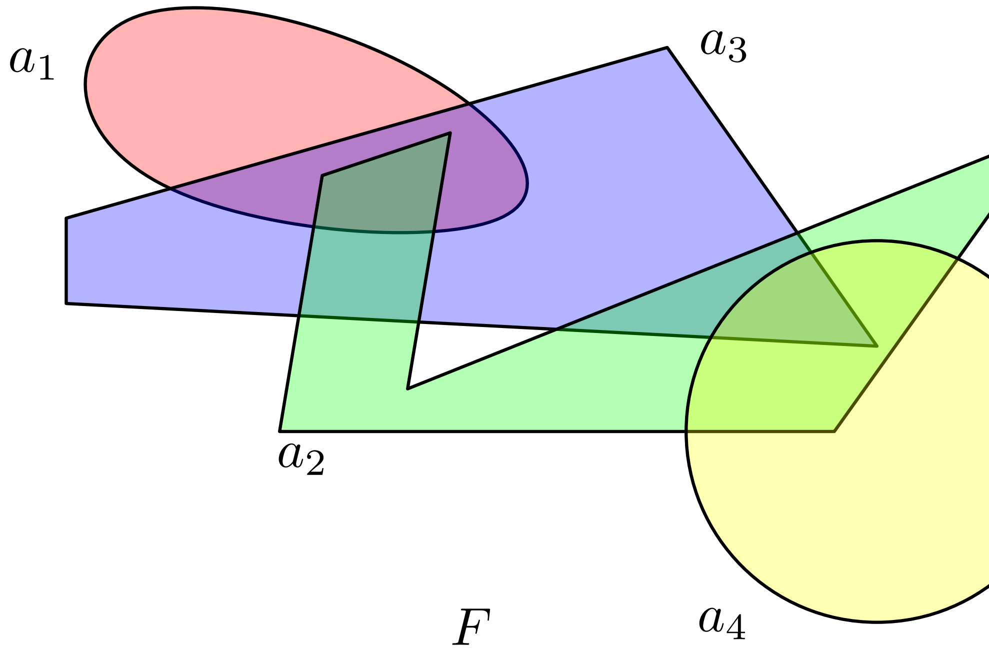
A subset of the selectors...
but permutations are easier to analyze than maps.



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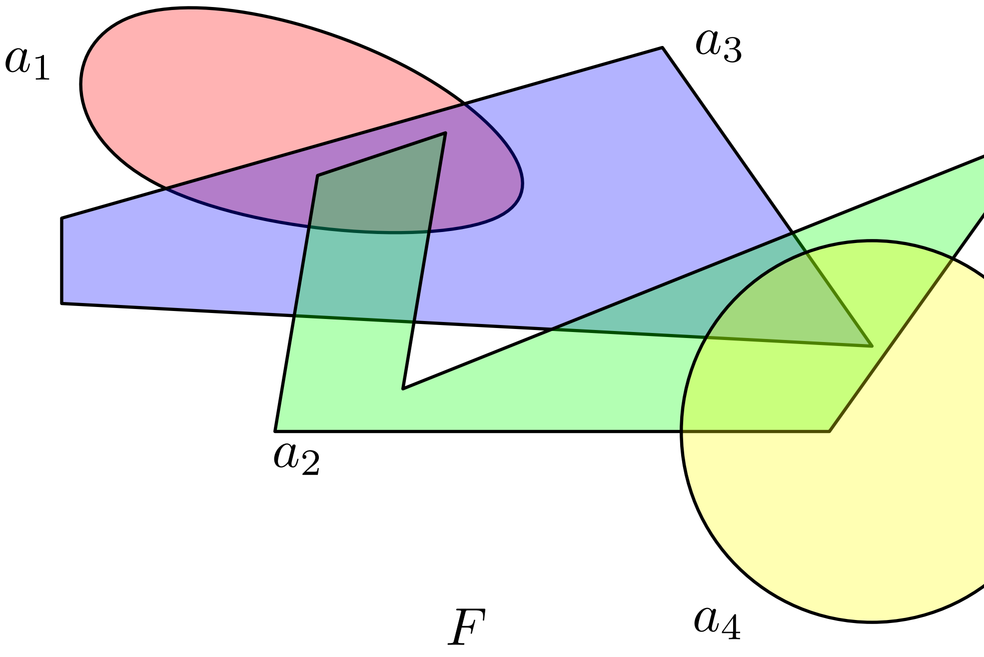

$\rho = (1, 3, 2, 4)$
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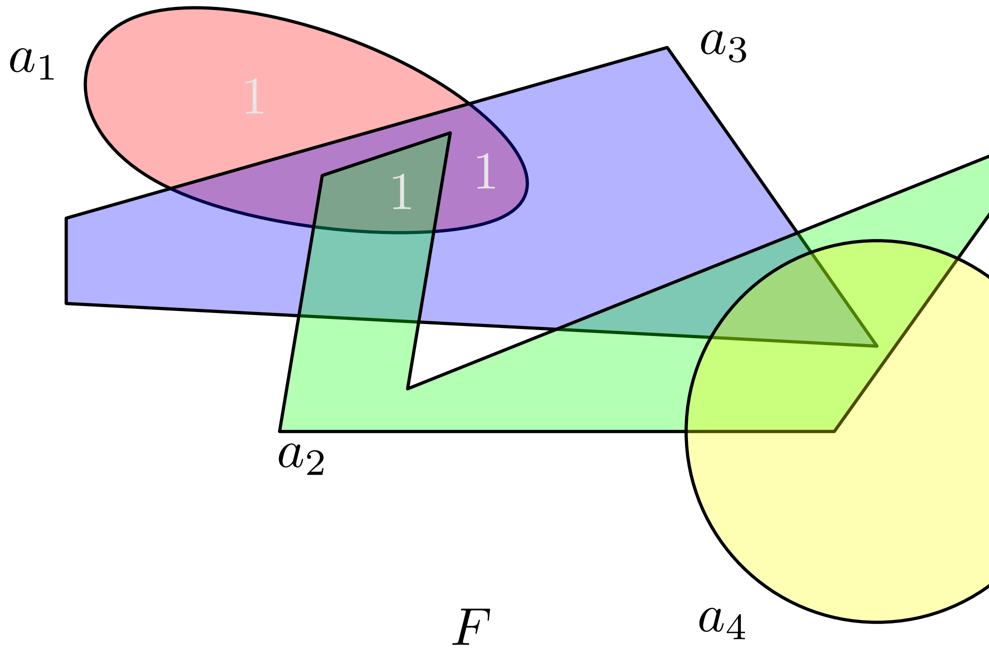

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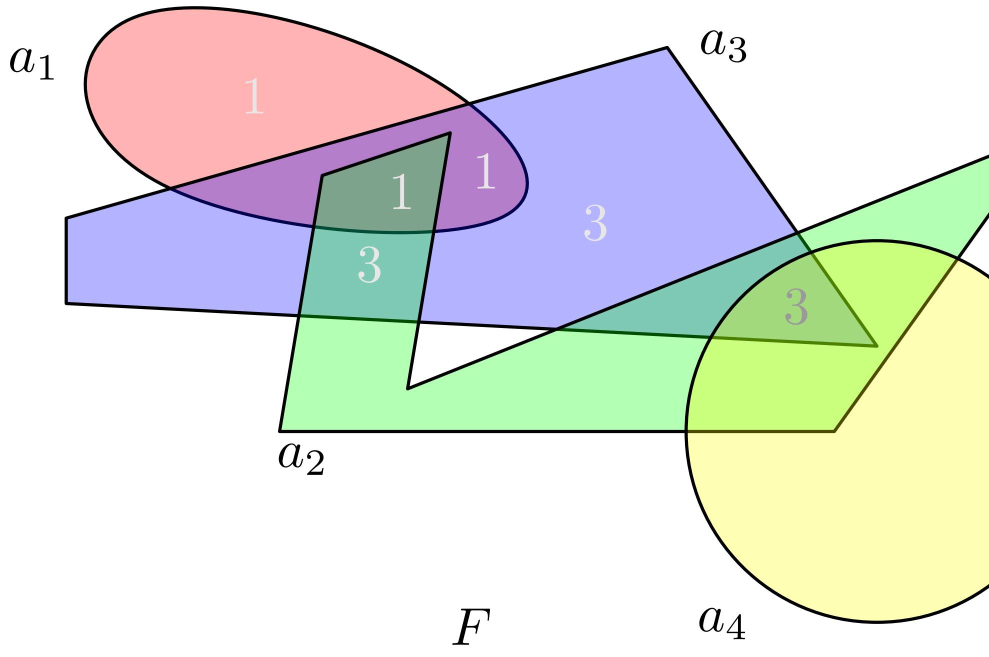

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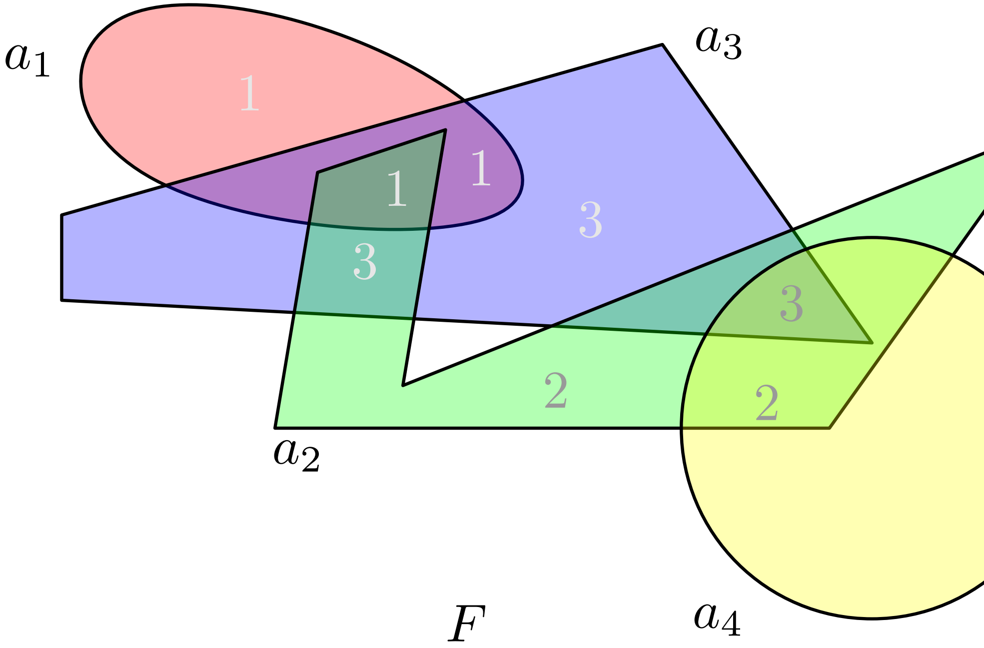

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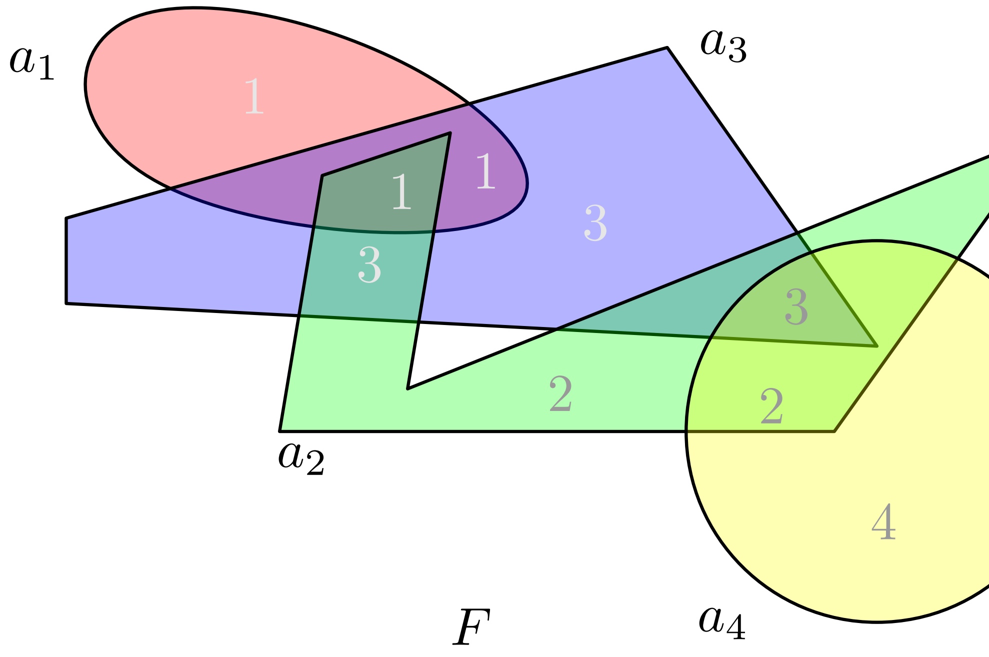

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Label $V(F) = \{\tau_1, \tau_2, \ldots, \tau_m\}$.

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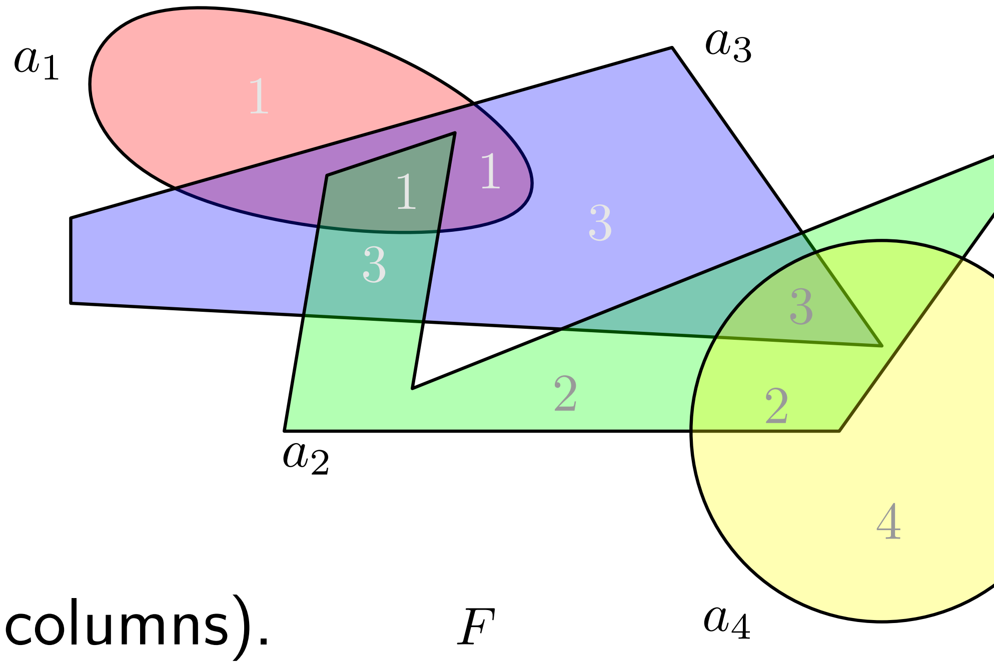

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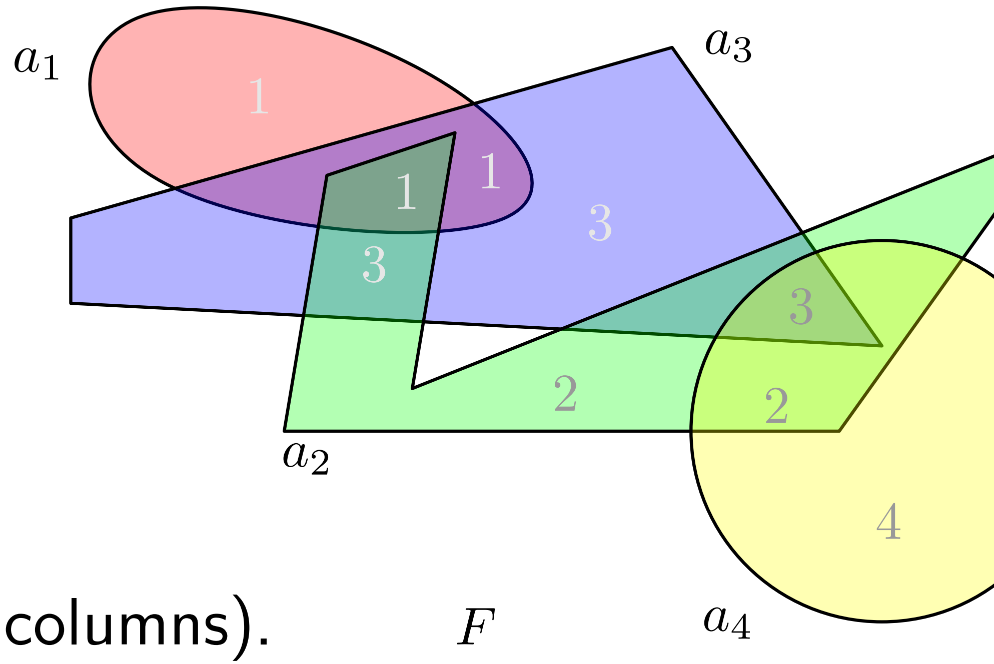

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$\Gamma_{i,j} = 1$ *if and only if* $j \in \tau_i$.

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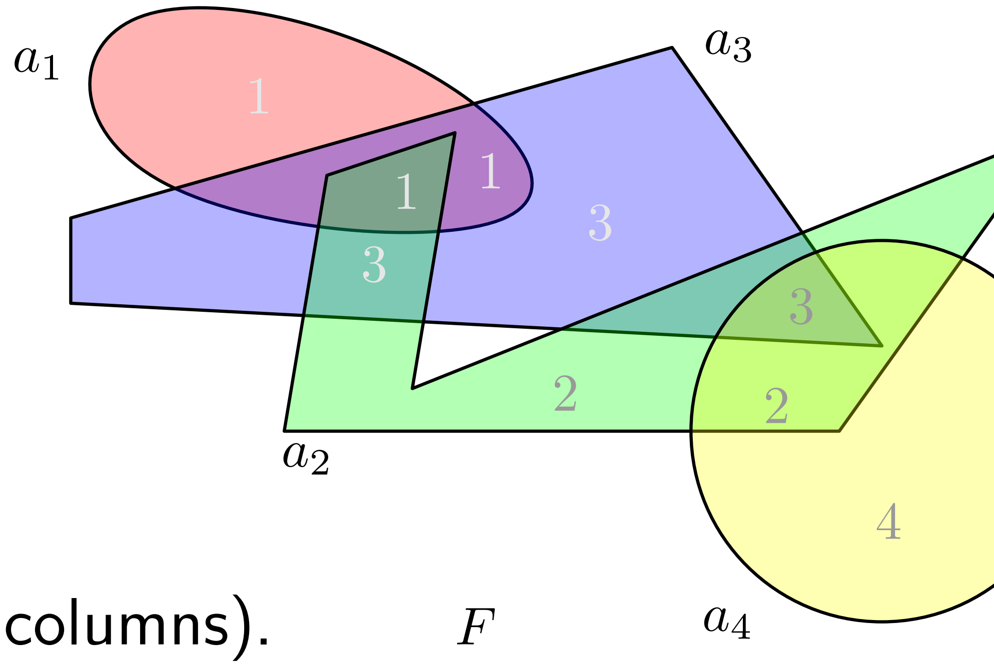

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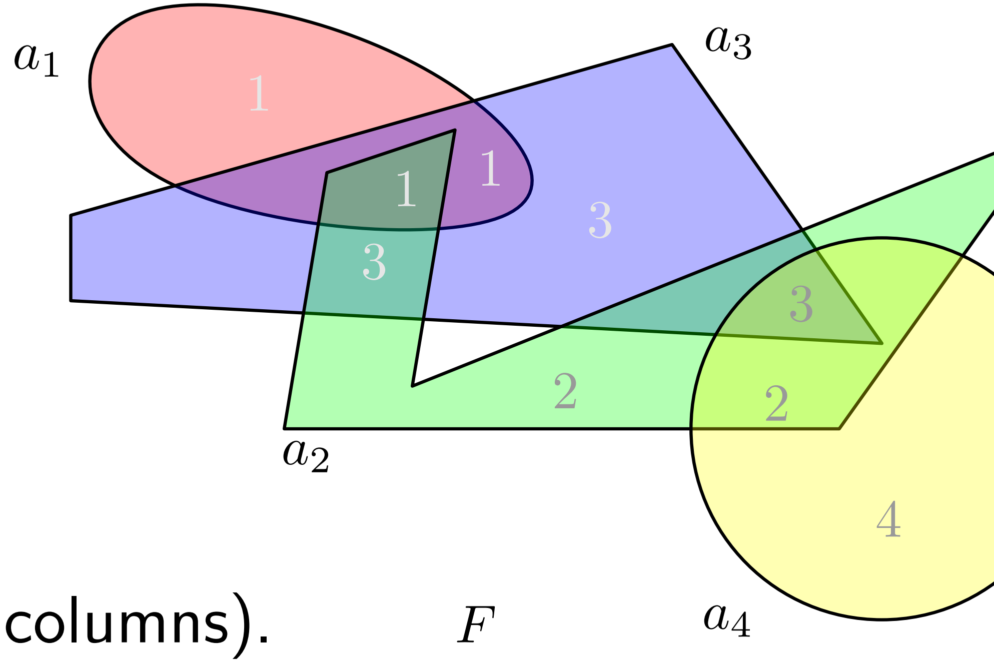

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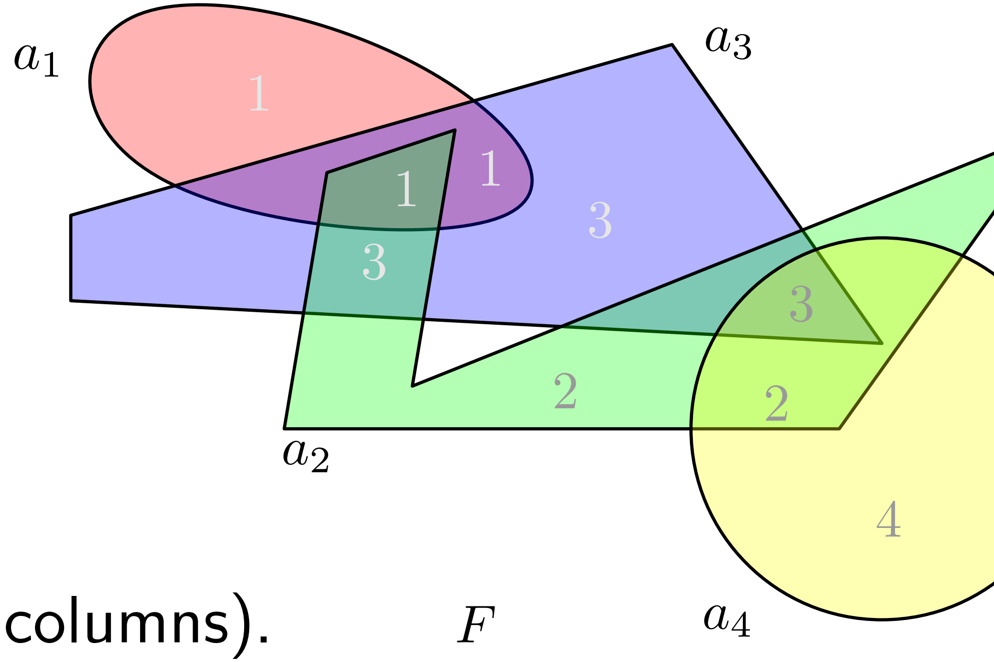

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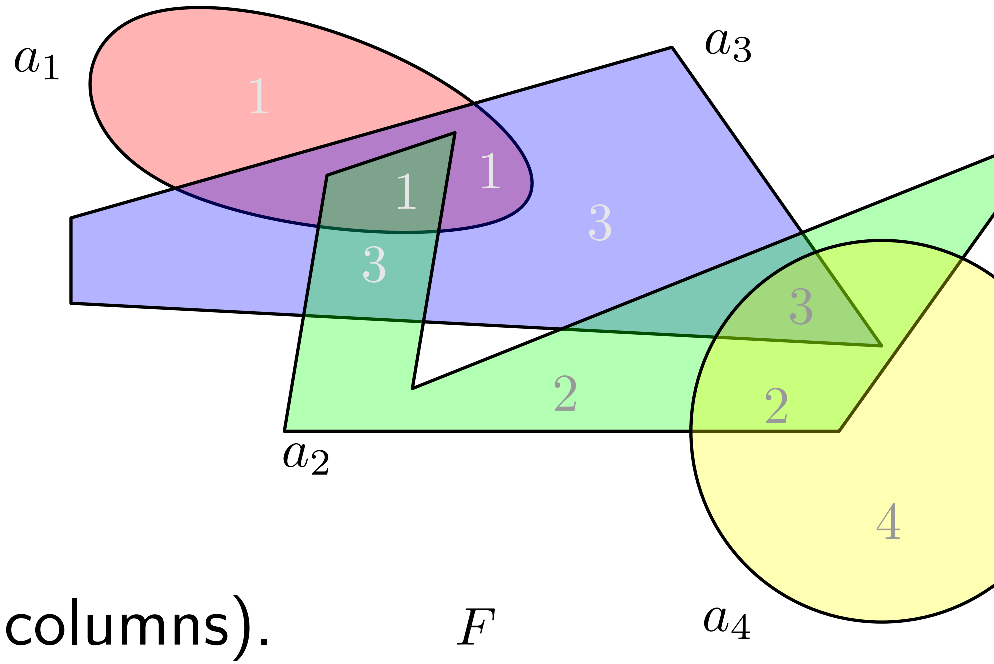

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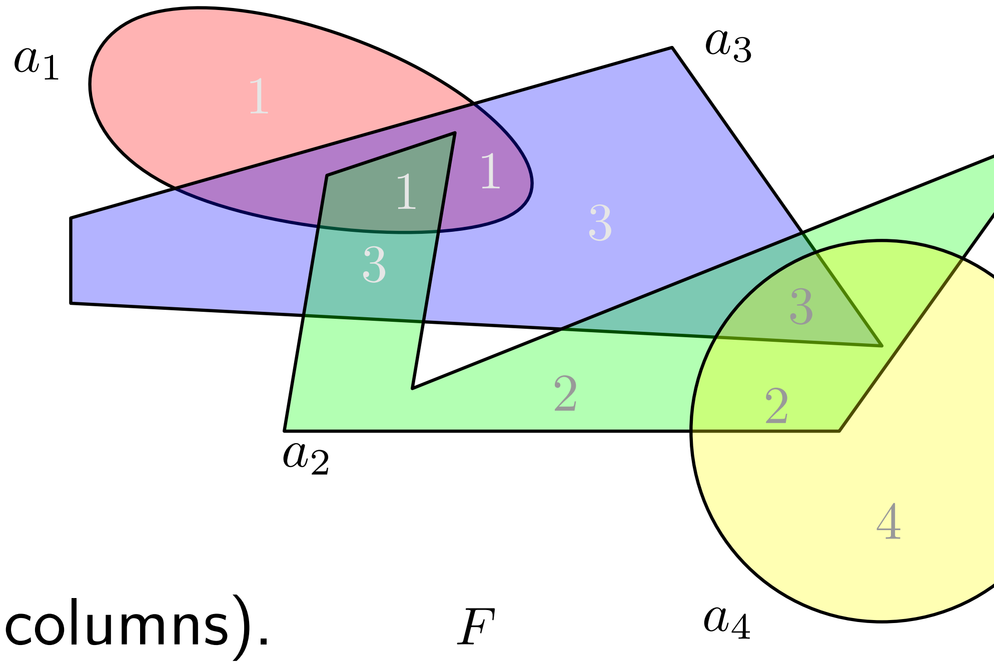

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		i_1	i_2	i_3		i_4		i_5	
$\vdots$									
j_3	$0 \cdots 0$	0	0	1	?	1	?	1	?
$\vdots$									
j_1	$0 \cdots 0$	1	1	1	?	1	?	1	?
j_2	$0 \cdots 0$	0	1	1	?	1	?	1	?
j_4	$0 \cdots 0$	0	0	0	$0 \cdots 0$	1	?	1	?
j_5	$0 \cdots 0$	0	0	0	$0 \cdots 0$	0	$0 \cdots 0$	1	?
$\vdots$									

$\rho \overset{\text{def}}{=}$ uniformly chosen **random** permutation of $[n]$.

$p(k) \overset{\text{def}}{=}$ probability that K_ρ contains at least one simplex of dimension k.

Proposition. [GMPST'15]
$$p\left(\lceil 2e \ln m \rceil \left\lceil 2 + \ln \frac{n}{\ln m} \right\rceil\right) \leq \frac{1}{2}.$$

#4. Zeta transform and its computation

SIAM J. COMPUT.
Vol. 39, No. 2, pp. 546–563

SET PARTITIONING VIA INCLUSION-EXCLUSION*

ANDREAS BJÖRKLUND†, THORE HUSFELDT†, AND MIKKO KOIVISTO‡

For $f : 2^{[n]} \to \mathbb{R}$ define $\widehat{f} : 2^{[n]} \to \mathbb{R}$ by

$$\widehat{f}(S) = \sum_{T \subseteq S} f(T).$$

$\widehat{f}$ is the **Zeta transform** of f.

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Suppose f is stored in an array.
Complexity of computing the array for $\widehat{f}$?

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Naive computation adds up to

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = (1+2)^n \text{ calls to } f.$$

$\rightsquigarrow O(3^n)$ time.

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Proof

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$$g_0(S) = f(S)$$

$$g_i(S) = \begin{cases} g_{i-1}(S) + g_{i-1}(S \setminus \{i\}) & \text{if } i \in S \\ g_{i-1}(S) & \text{otherwise} \end{cases}$$

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We have $$g_i(S) = \sum_{\substack{T \subseteq S \\ T \cap \{i+1,\dots,n\} = S \cap \{i+1,\dots,n\}}} f(T)$$

For $f : 2^{[n]} \to \mathbb{R}$ define $\widehat{f} : 2^{[n]} \to \mathbb{R}$ by

$$\widehat{f}(S) = \sum_{T \subseteq S} f(T).$$

$\widehat{f}$ is the **Zeta transform** of f.

Suppose f is stored in an array.
Complexity of computing the array for $\widehat{f}$?

Naive computation adds up to

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = (1+2)^n \text{ calls to } f.$$

$\rightsquigarrow O(3^n)$ time.

Proposition. [Yates'37]
$\widehat{f}$ can be computed in $O(2^n n)$ time.

Dynamic programming.

Proof

For $i \in [n]$ and $S \subseteq [n]$ define $g_i(S)$ by

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$$g_n = \widehat{f}.$$

#5. Graph coloring via inclusion-exclusion

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SET PARTITIONING VIA INCLUSION-EXCLUSION*

ANDREAS BJÖRKLUND†, THORE HUSFELDT†, AND MIKKO KOIVISTO‡

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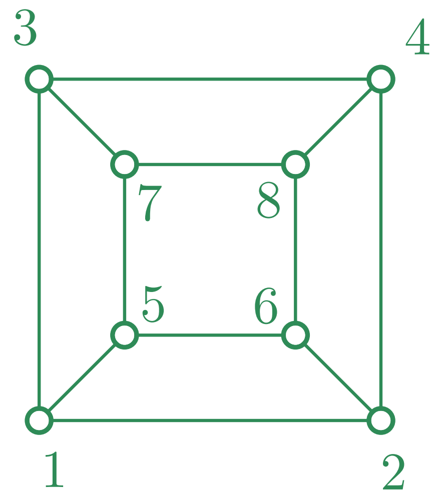

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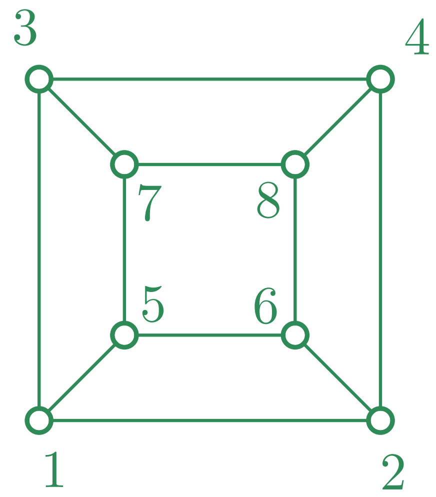

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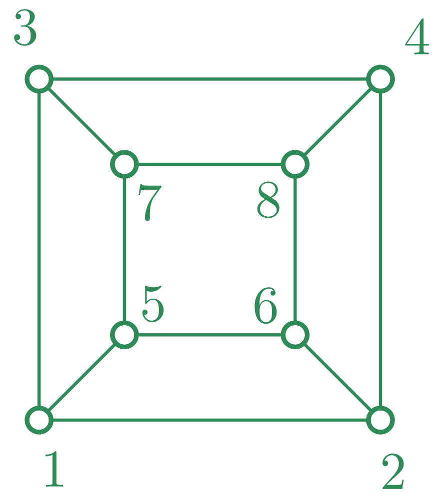

$\{1, 4, 6\}$ and $\{1, 8\}$ are in $\mathcal{I}$

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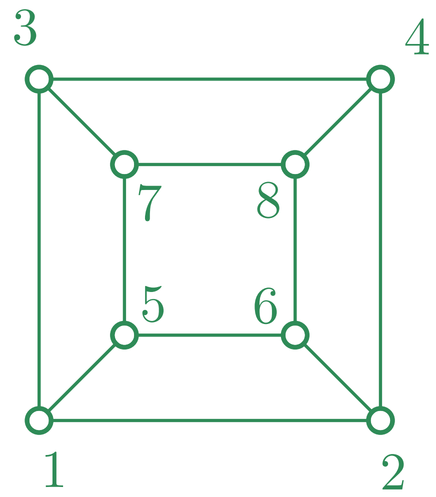

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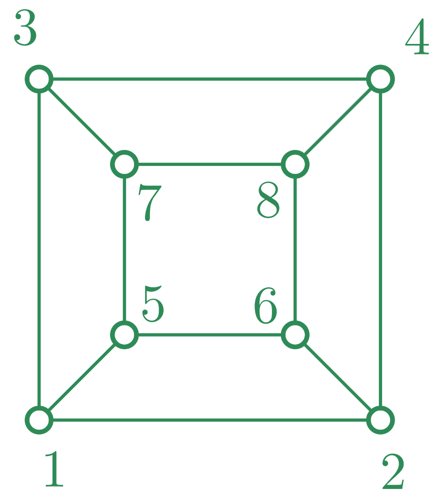

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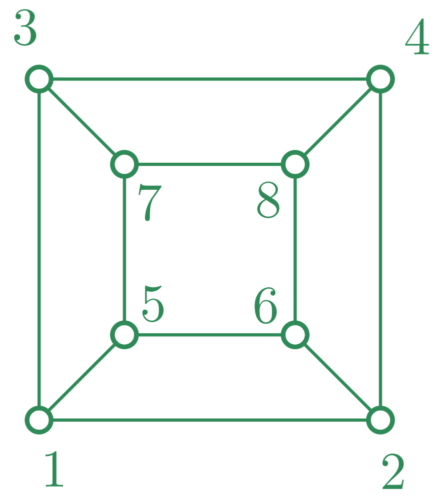

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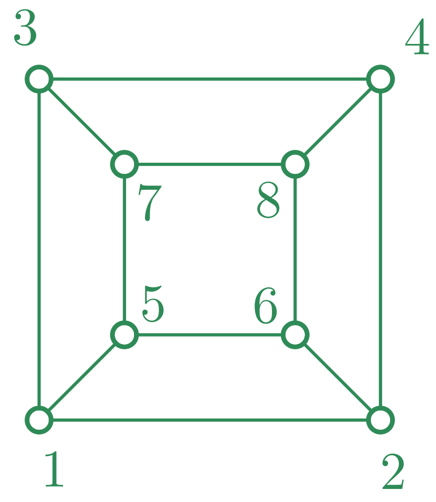

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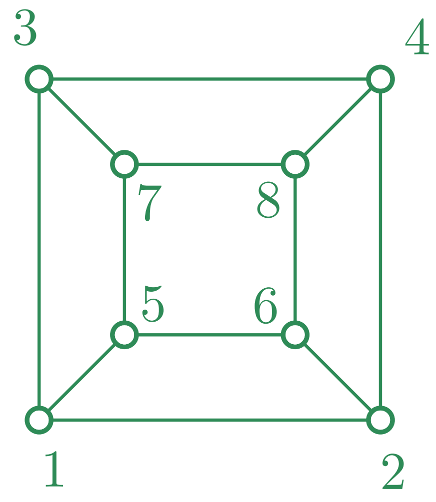

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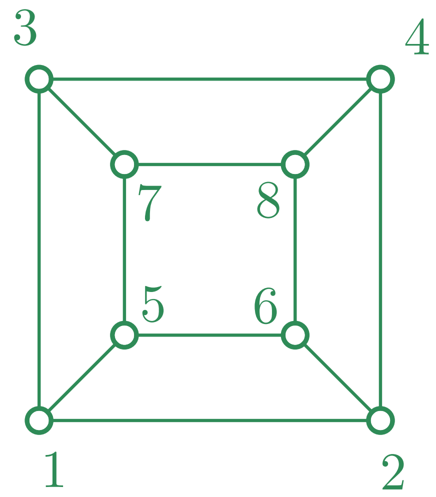

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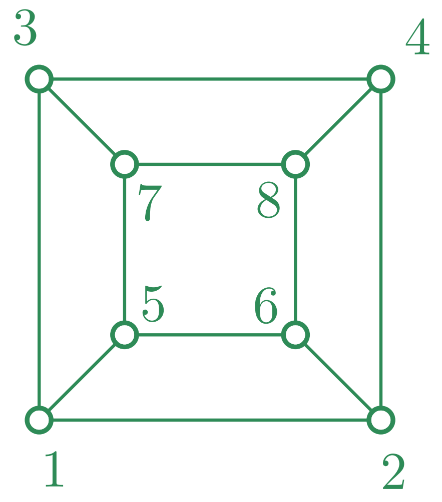

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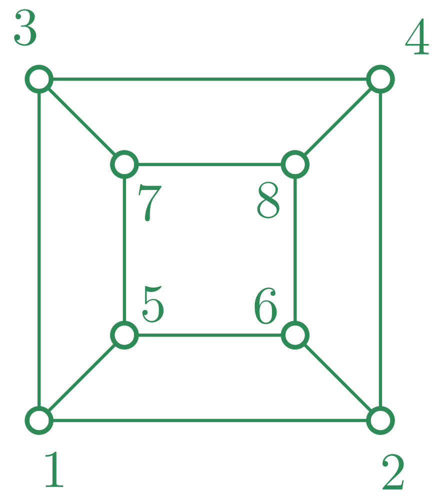

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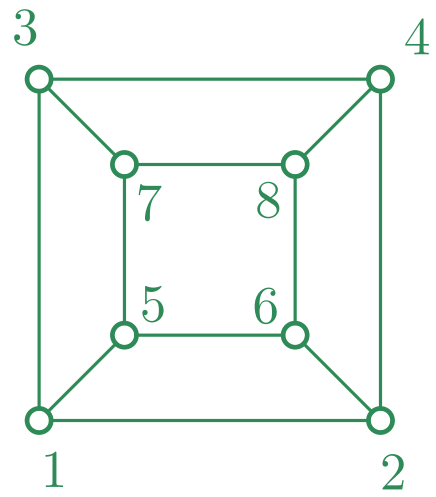

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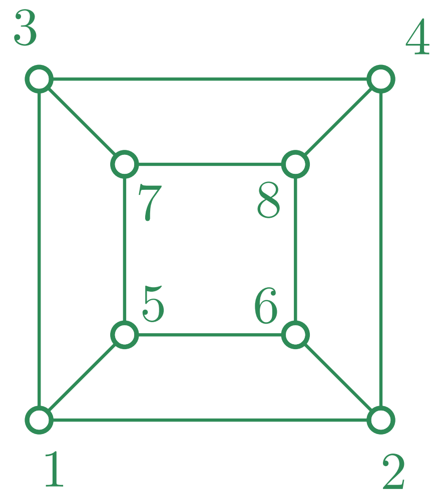

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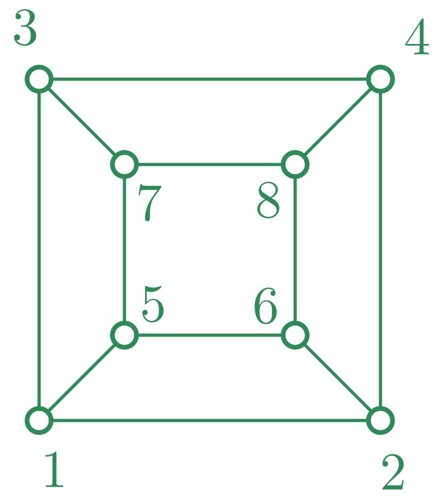

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#6. What if... ?

Could we solve **graph k-coloring** using **simplified** inclusion-exclusion formulas?

1. Start with a graph G on n vertices.
2. Consider the set system $F = \{A_1, A_2, \ldots, A_n\}$

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$O(n12^{n/4})$ for $k = 2$ [Madsen-Nielsen-Skerjnaa'02]

Thank you for your attention!